Network Algorithms - Fall 2008
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Handout #5: Minimum Cuts

Deadline: Wednesday 15/12/08.

[CCPS] refers to the book "Combinatorial Optimization" by Cook, Cunningham, Pulleyblank, and Schrijver.

[Hu] refers to the book "Integer Programming and Network Flows", by T.C. Hu.

1. Prove or refute:
   (a) If v is a node of minimum degree, then there exists a legal ordering $\{v_i\}_{i=1}^n$ in which $v = v_n$.
   (b) If $|V| > 2$, then there exists at least two distinct pairs of vertices (a_1, b_1) and (a_2, b_2) such that $\lambda(G; a_i, b_i) = u(\delta(\{a_i\}))$.
2. ([CCPS, question 3.54]) Let $p, q, r \in V$ denote three distinct vertices. Prove that if $\lambda(G; p, q) \leq \lambda(G; p, r) \leq \lambda(G; r, q)$, then $\lambda(G; p, q) = \lambda(G; p, r)$.
3. ([CCPS, question 3.56] A cut $\delta(A)$ is *minimal* if there does not exist a set $B \neq A, V \setminus A$ such that $\delta(B) \subsetneq \delta(A)$. Prove that the random contraction algorithm always returns a minimal cut.
4. ([CCPS], questions 3.60 and 3.61) Prove an upper bound on the number of different minimum global cuts a graph may have. Is this bound tight? (I.e. can you present a graph that contains so many minimum global cuts?)
5. Prove a an upper bound on the number of min-cuts in a *directed* graph. Is your bound tight?!
6. ([CCPS], question 3.62) How many times does one need to run the random contraction algorithm so that the probability that all minimum global cuts are found is at least $1 - \frac{1}{n}$?
7. ([CCPS], questions 3.64 and 3.65) Assume that the random contraction algorithm is run for $n - 2\alpha$ iterations. After that, a cut $\delta(A)$ is picked uniformly from the remaining graph. What is the probability that this cut is an α-approximate minimum cut? (I.e. prove a lower bound on $Prob[u(\delta(A)) \leq \alpha \cdot \lambda(G)]$.)
8. ([CCPS], question 3.66) Let $G = (V, E)$ denote an undirected graph with edge capacities $u : E \to \mathbb{R}^{\geq 0}$. Show that
$$u(\delta(A)) + u(\delta(B)) \geq u(\delta(A \cup B)) + u(\delta(A \cap B))$$
for every two subsets $A, B \subseteq V$.