Network Algorithms - Fall 2002
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Handout #4: Min-cost flow

Deadline: Wednesday 26/11/02.

1. The min-cost circulation problem is defined as follows:

   input: (i) A directed graph $G = (V, E)$ with a symmetric edge set (i.e. $(u, v) \in E \Rightarrow (v, u) \in E$). (ii) Edge capacities $u : E \longrightarrow \mathbb{R}$. (ii) Edge costs $c : E \longrightarrow \mathbb{R}$ that are anti-symmetric (i.e. $c(u, v) = -c(v, u)$).

   output: A circulation $f : E \longrightarrow \mathbb{R}$ that satisfies:

   (a) Edge capacities: $\forall e \in E : \quad f(e) \leq u(e)$.
   (b) Anti-symmetry: $\forall (a, b) \in E : \quad f(a, b) = -f(b, a)$.
   (c) Flow conservation: $\forall v \in V : \quad \sum_{\{u:(u,v)\in E\}} f(u, v) = 0$.

   goal: Minimize the cost of the circulation defined by:

   $$c \cdot f \triangleq \sum_{e \in E} c(e) \cdot f(e).$$

   Reduce the following problems to the min-cost circulation problem.

   (a) **Min cost-flow**: Each vertex has a demand/supply amount $b(v)$. A positive $b(v)$ means that v is a sink with demand $b(v)$, a negative $b(v)$ means that v is a source with supply amount $b(v)$. The goal is to solve the following linear program:

   $$\text{Minimize } \sum_{e} c(e) \cdot f(e)$$

   subject to

   $$\forall e \in E : 0 \leq f(e) \leq u(e)$$
   $$\forall v \in V : \sum_{\{u:(u,v)\in E\}} f(u, v) - \sum_{\{u:(v,u)\in E\}} f(v, u) = b(v).$$

   (b) **Capacitated Transportation**: Here the graph $G = (P \cup Q, E)$ is a bipartite graph. Each vertex $p \in P$ has a supply amount $a(p)$. Each vertex $q \in Q$ has a demand amount $b(q)$. Each edge (p, q) has a cost $c(p, q)$ and a capacity $u(p,q)$. The problem is:

$$\begin{aligned}
&\text{Minimize } \sum_e c(e)\cdot f(e)\\
&\text{subject to}\\
&\ \forall e\in E: 0\le f(e)\le u(e)\\
&\ \forall p\in P: \sum_{\{q:(p,q)\in E\}} f(p,q)=a(p)\\
&\ \forall q\in Q: \sum_{\{p:(p,q)\in E\}} f(p,q)=b(q).
\end{aligned}$$

(c) **Min-cost flow with supply ranges and flow ranges**:

$$\begin{aligned}
&\text{Minimize } \sum_e c(e)\cdot f(e)\\
&\text{subject to}\\
&\ \forall e\in E: \ell(e)\le f(e)\le u(e)\\
&\ \forall v\in V: a(v)\le \sum_{\{u:(u,v)\in E\}} f(u,v)-\sum_{\{u:(v,u)\in E\}} f(v,u)\le b(v).
\end{aligned}$$

Note that we allow $\ell(e)=-\infty$ and $u(e)=\infty$.

(d) **Bipartite Matching**: Let $G=(A\cup B,E)$ denote a bipartite graph with edge weights $w(e)$.

i. Find a maximum cardinality matching $M\subseteq E$.

ii. Find a maximum weight matching $M\subseteq E$.

iii. Find a maximum weight matching $M\subseteq E$ of cardinality k.

2. Suppose that in every iteration of the negative cycle cancellation algorithm for min-cost flow a cycle Γ is chosen such that

$$\forall \text{ cycle } \Gamma' \text{ in } G_f: \quad \delta(\Gamma)\cdot c(\Gamma)\le \delta(\Gamma')\cdot c(\Gamma'),$$

where $\delta(\Gamma)=\min_{e\in\Gamma} u_f(e)$.

Assume that the edge capacities and costs are integers. Prove that the number of iterations of this cycle canceling algorithm is polynomial. (Can you show that finding a cycle that maximizes the improvement is NP-Hard?)