Network Algorithms - Fall 2002
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Handout #5: Applications of min-cost flow

Deadline: Wednesday 04/12/02.

Solve the following problems by reducing them to your favorite min-cost flow problem.

1. **More general capacitated transportation**: We are given a set P of car producers and a set Q of car dealers. There are k car models $m_1, \ldots, m_k$. Each dealer $q \in Q$ has a demand d_i for car model m_i. Each producer $p \in P$ has a production capacity u_i for car model m_i. The cost of shipping a car of model m_i from p to q is $c(p, q, i)$.

   The goal is find a min-cost solution to the following problem: (i) how many cars of each model should each producer produce? (ii) how many cars of each model should each producer send to each dealer?

2. **Reconstructing a binary matrix**: Consider a binary matrix $A = \{a_{ij}\}_{i,j=1,\ldots,n}$. Assume that for every entry a_{ij} we are given a probability p_{ij} that $a_{ij} = 1$. Assume that we are given the column and row sums:

$$r_i = \sum_j a_{ij}$$
$$c_j = \sum_i a_{ij}.$$

   The goal is to reconstruct the most likely matrix A from the row and column sums given the probabilities p_{ij}.

3. **Optimal loading of an airplane**: An airplane travels starting from city 1 to city $i+1$ until it reaches the final city n. The number of passengers it can carry is bounded by p. Let b_{ij} denote the number of passengers who are willing to travel from city i to city j. Let f_{ij} denote the price of a ticket from city i to city j.

   Our goal is to compute an assignment $\{n_{ij}\}_{ij}$, where n_{ij} denotes the number of passengers that the plane carries from city i to city j so that the total revenue is maximized.

4. **Linear programs with consecutive 1's in columns**.

Solve the linear program

$$\text{Minimize } \sum_i c_i \cdot x_i$$

subject to

$$A \cdot \vec{x} \geq \vec{b},$$
$$\vec{x} \geq \vec{0},$$

Where A is a binary matrix where each column is a binary word of the form $0^*1^*0^*$. (Note that $A \cdot \vec{x} \geq \vec{b}$ means that the inner product of the ith row of A with $\vec{x}$ is greater than or equal to b_i.)

Suggestion: construct a small matrix A and try to apply the reduction to the example.

5. **Car replacement problem**: The total cost associated with buying a car in year i and selling it in year j is c_{ij} (this is the cost of buying it plus the cost of maintaining it, minus the cost of selling it). We would like to own at least one car at all times. Determine the cheapest schedule of buying and selling cars.

6. Prove the Min-Flow Max-Cut Theorem. Suppose we have a flow problem with lower bounds $\ell(e)$ and upper bounds $u(e)$. Our goal is to find a legal (s,t)-flow f (i.e. $\ell(e) \leq f(e) \leq u(e)$) with the smallest amount of flow.

   (a) Prove that:

   $$\min |f| = \max_{S \subset V : s \in S, t \notin S} \ell(\delta(S)) - u(\delta(V \setminus S)).$$

   (b) Reduce the min flow problem to a max flow problem.