Network Algorithms - Fall 2002
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Handout #7: Polyhedra and extreme points

Deadline: Monday 30/12/02.

(Questions 2-4 are from the book of Bertsimas & Tsitsiklis.)

1. Let A denote an $m \times m$ matrix, C denote an $m \times (n-m)$ matrix, I_{n-m} denote the identity $(n-m) \times (n-m)$ matrix, and O denote the all zeros $(n-m) \times m$ matrix. Let X denote the matrix with the following blocks:

$$X = \begin{pmatrix} C & A \\ I_{n-m} & O \end{pmatrix}.$$

   Prove: if $rank(X) = n$, then $rank(A) = m$.

2. We know the following facts:

   - Every linear program can be reduced to an equivalent linear program in standard form. (Can you define what equivalence of linear programs means here?)
   - If P is a non-empty polyhedron represented in standard form, then P contains at least one extreme point.

   What is wrong with the following conclusion? If P is a non-empty polyhedron, then it contains at least one extreme point.

3. (extreme points of isomorphic polyhedra) An *affine transformation* is a mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ that is defined by

$$f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b},$$

   where $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{b}$ is a vector.

   Let $P \subset \mathbb{R}^n$ and Let $Q \subset \mathbb{R}^m$ denote two polyhedra. We say that P and Q are *isomorphic* if there exist affine transformations $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^n$, such that

$$g(f(\mathbf{x})) = \mathbf{x} \text{ for every } x \in P$$
$$f(g(\mathbf{y})) = \mathbf{y} \text{ for every } y \in Q.$$

   (a) Show that there is a one-to-one correspondence between the extreme points of P and the extreme points of Q. (Hint: show that if $x \in P$ is an extreme point, then so is $f(x) \in Q$.)

(b) Prove that introducing slack variables leads to an isomorphic polyhedron. Formally, let

$$P = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \geq \mathbf{b},\ \ \mathbf{x} \geq \mathbf{0}\},$$

where $\mathbf{A}$ is an $m \times n$ matrix. Define Q as follows:

$$Q = \left\{(\mathbf{x}, \mathbf{z}) \in \mathbf{R}^{n+m} \mid \mathbf{A}\mathbf{x} - \mathbf{z} = \mathbf{b},\ \ \mathbf{x} \geq \mathbf{0},\ \ \mathbf{z} \geq \mathbf{0}\right\},$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{z} \in \mathbb{R}^m$.

Prove that P and Q are isomorphic.

4. (Carathéodory's theorem) Let $\mathbf{A}_1, \ldots, \mathbf{A}_n$ be a collection of vectors in $\mathbb{R}^m$.

(a) The cone spanned by $\{\mathbf{A}_i\}_{i=1}^n$ is defined by:

$$C = \left\{\sum_i \lambda_i \cdot \mathbf{A}_i \mid \lambda_1, \ldots, \lambda_n \geq 0\right\}.$$

Show that every $\mathbf{x} \in C$ can be expressed as the linear combination of at most m vectors in the set $\{\mathbf{A}_i\}_{i=1}^n$. Moreover, show that in this linear combination only positive scalars can be used. Hint: Given a vector $\mathbf{b} \in C$, consider the polyhedron:

$$P_{\mathbf{b}} = \{\lambda \in \mathbb{R}^n \mid \mathbf{A}\lambda = \mathbf{b},\ \ \lambda \geq \mathbf{0}\}$$

(b) Let P denote the convex hull of the vectors $\{\mathbf{A}_i\}_{i=1}^n$. Namely,

$$C = \left\{\sum_i \lambda_i \cdot \mathbf{A}_i \mid \sum_i \lambda_i = 1,\ \ \lambda_1, \ldots, \lambda_n \geq 0\right\}.$$

Show that every $\mathbf{x} \in C$ can be expressed as the convex combination of at most $m + 1$ vectors in the set $\{\mathbf{A}_i\}_{i=1}^n$.