Network Algorithms - Fall 2006
Dr. Guy Even
`http://hyde.eng.tau.ac.il/CO06/`

Solution session: Sunday 10/12/04.

[AMO] refer to the book "Network Flows" by Ahuja, Magnanti, and Orlin.

[CCPS] refers to the book "Combinatorial Optimization" by Cook, Cunningham, Pulleyblank, and Schrijver.

[E] refers to the book "Graph Algorithms" by Shimon Even.

Handout #3: Applications of maximum flow, Push-Relabel max flow alg, Applications of min cost flow

1. (Erdős-Szekeres Theorem) Let $A = (a_1, a_2, \ldots, a_n)$ denote a sequence of n different real numbers. Prove that if $n \geq rs + 1$, then A has an increasing subsequence of length $s + 1$ or a decreasing subsequence of length $r + 1$.

2. Devise an algorithm that finds a minimum capacity cut $\delta(R)$ with the fewest number of edges. Namely,

$$u(\delta(R)) = \min\{u(\delta(R')) : \delta(R') \text{ is a minimum capacity cut}\}$$
$$|\delta(R)| = \min\{|\delta(R')| : \delta(R') \text{ is a minimum capacity cut}\}.$$

3. Answer exercise 7.8 in [AMO]: A *least vital edge* is an edge whose deletion causes the least decrease in in the maximum flow value. Prove or refute the following statements (flow here is defined in the standard fashion, not the anti-symmetric definition).

   (a) If f is a maximum flow and $f(e) = 0$ then e is a least vital edge.

   (b) If f is a maximum flow and $f(e) = \min\{f(e') : e' \in E\}$, then e is a least vital edge.

   (c) If e belongs to a minimum capacity cut, then e is not a least vital edge.

4. Consider the following modification of the generic push-relabel max flow algorithm. The main loop picks an active node v (i.e., $e(v) > 0$) whose label $d(v)$ is maximum.

   (a) Define the potential function $\varphi = \max\{d(v) \mid v \in V \;\&\; e(v) > 0\}$. Bound the maximum number of nonsaturating push operations. Hint: bound the number of nonsaturating push operations that can be executed while the potential does not change.

   (b) Suggest an implementation of the push/relabel algorithm based on this modification with a running time of $O(n^3)$.

5. **More general capacitated transportation**: We are given a set P of car producers and a set Q of car dealers. There are k car models $m_1, \ldots, m_k$. Each dealer $q \in Q$ has a demand d_i for car model m_i. Each producer $p \in P$ has a production

capacity u_i for car model m_i. The cost of shipping a car of model m_i from p to q is $c(p, q, i)$.

The goal is find a min-cost solution to the following problem: (i) how many cars of each model should each producer produce? (ii) how many cars of each model should each producer send to each dealer?

6. **Reconstructing a binary matrix**: Consider a binary matrix $A = \{a_{ij}\}_{i,j=1,\ldots,n}$. Assume that for every entry a_{ij} we are given a probability p_{ij} that $a_{ij} = 1$. Assume that we are given the column and row sums:

$$r_i = \sum_j a_{ij}$$
$$c_j = \sum_i a_{ij}.$$

The goal is to reconstruct the most likely matrix A from the row and column sums given the probabilities p_{ij}.

7. **Optimal loading of an airplane**: An airplane travels starting from city 1 to city $i+1$ until it reaches the final city n. The number of passengers it can carry is bounded by p. Let b_{ij} denote the number of passengers who are willing to travel from city i to city j. Let f_{ij} denote the price of a ticket from city i to city j.

Our goal is to compute an assignment $\{n_{ij}\}_{ij}$, where n_{ij} denotes the number of passengers that the plane carries from city i to city j so that the total revenue is maximized.

8. **Linear programs with consecutive 1's in columns**.

Solve the linear program

$$\text{Minimize } \sum_i c_i \cdot x_i$$

subject to

$$A \cdot \vec{x} \geq \vec{b},$$
$$\vec{x} \geq \vec{0},$$

Where A is a binary matrix where each column is a binary word of the form 0*1*0*. (Note that $A \cdot \vec{x} \geq \vec{b}$ means that the inner product of the ith row of A with $\vec{x}$ is greater than or equal to b_i.)

Suggestion: construct a small matrix A and try to apply the reduction to the example.

9. **Car replacement problem**: The total cost associated with buying a car in year i and selling it in year j is c_{ij} (this is the cost of buying it plus the cost of maintaining it, minus the cost of selling it). We would like to own at least one car at all times. Determine the cheapest schedule of buying and selling cars.