example: longest paths in DAGs

paths ending in $V_{10}$

$\text{length} = 5$

$\text{length} = 1$
We denote the length of a path \( \Gamma \) by \( |\Gamma| \).

**Definition**

A path \( \Gamma \) that ends in vertex \( v \) is a longest path ending in \( v \) if \( |\Gamma'| \leq |\Gamma| \) for every path \( \Gamma' \) that ends in \( v \).

Note: there may be multiple longest paths ending in \( v \) (hence “a longest path” rather than “the longest path”).

**Definition**

A path \( \Gamma \) is a longest path in \( G \) if \( |\Gamma'| \leq |\Gamma| \), for every path \( \Gamma' \) in \( G \).

**Question**

Does a longest path always exist in a directed graph?
If a directed graph has a cycle, then there does not exist a longest path. Indeed, one could walk around the cycle forever. However, longest paths do exist in DAGs.

**Lemma**

If $G = (V, E)$ is a DAG, then there exists a longest path that ends in $v$, for every $v$. In addition, there exists a longest path in $G$.

Proof: The length of every path in a DAG is at most $|V| - 1$. [Or, every path is simple, hence, the number of paths is finite.]
a → b → c → d → a → b → c → d → .....
Goal: compute, for every $v$ in a DAG, a longest path that ends in $v$. We begin with the simpler task of computing the length of a longest path.

**Specification**

Algorithm *longest-path* is specified as follows.

- **input:** A DAG $G = (V, E)$.
- **output:** A delay function $d : V \rightarrow \mathbb{N}$.
- **functionality:** For every vertex $v \in V$: $d(v)$ equals the length of a longest path that ends in $v$.

Application: Model circuits by DAGs. Assume all gates complete their computation in one unit of time. The delay of the output of a gate $v$ equals $d(v)$.
example: delay function
Algorithm 2 longest-path-lengths($V, E$) - An algorithm for computing the lengths of longest paths in a DAG. Returns a delay function $d(v)$.

1. topological sort: $(v_0, \ldots, v_{n-1}) \leftarrow TS(V, E)$.
2. For $j = 0$ to $(n - 1)$ do
   1. If $v_j$ is a source then $d(v_j) \leftarrow 0$.
   2. Else

   $$d(v_j) = 1 + \max \left\{ d(v_i) \mid i < j \text{ and } (v_i, v_j) \in E \right\}.$$ 

One could design a “single pass” algorithm; the two pass algorithm is easier to prove.
\[ d(v_0) = 0 \]
\[ d(v_1) = 1 \]
\[ d(v_2) = 2 \]
\[ d(v_3) = 3 \]

\[ o(l(v_3)) = 1 + \max\{d(v_2), d(v_0)\} \]
\[ = 1 + \max\{2, 0\} \]
\[ = 1 + 2 = 3 \]
Let
\[ d(v) \triangleq \text{output of algorithm} \]
\[ \delta(v) \triangleq \text{the length of a longest path that ends in } v \]

**Theorem**

*Algorithm correct:* \( \forall j : d(v_j) = \delta(v_j) \).

**Proof:** Complete induction on \( j \). Basis for sources easy.

*ind. hyp.:* \( \forall i \leq j : d(v_i) = \delta(v_i) \)

*step:* prove that \( d(v_{j+1}) = \delta(v_{j+1}) \)
We prove now that

1. \( \delta(v_{j+1}) \geq d(v_{j+1}) \), namely, there exists a path \( \Gamma \) that ends in \( v \) such that \( |\Gamma| \geq d(v_{j+1}) \).

2. \( \delta(v_{j+1}) \leq d(v_{j+1}) \), namely, for every path \( \Gamma \) that ends in \( v \) we have \( |\Gamma| \leq d(v_{j+1}) \).
length of longest path ending in \( v_{j+1} \) \[ \delta(v_{j+1}) \geq d(v_{j+1}) \] output of alg. longest-path

need to show \( \exists \text{ path } \Gamma \rightarrow v_{j+1} : |\Gamma| \geq d(v_{j+1}) \)

cases:
(1) \( v_{j+1} \) is a source. easy

(2) \( v_{j+1} \) is not a source. assume \( \deg_{\text{in}}(v_{j+1}) = 3 \)

\[ d(v_{j+1}) = 1 + \max \{ d(v_{i_1}), d(v_{i_2}), d(v_{i_3}) \} \]
(2) $v_{j+1}$ is not a source, assume $\deg_{in}(v_{j+1}) = 3$

$\begin{align*}
d(v_{j+1}) &= 1 + \max\{d(v_{i_1}), d(v_{i_2}), d(v_{i_3})\} \\

\text{assume:} & \\
d(v_{i_2}) &\geq \max\{d(v_{i_1}), d(v_{i_2}), d(v_{i_3})\} \\
\text{so:} & \\
d(v_{j+1}) &= 1 + d(v_{i_2})
\end{align*}$

Topo. Sort $\Rightarrow i_2 < j+1$.

ind. hyp. $\Rightarrow \exists$ path $\xrightarrow{T'} v_{i_2}: |T'| \geq d(v_{i_2})$

consider $T'$ extended by edge $v_{i_2} \rightarrow v_{j+1}$:

$|T' \circ (v_{i_2}, v_{j+1})| = |T'| + 1 \geq d(v_{i_2}) + 1 = d(v_{j+1})$

path ending in $v_{j+1}$

$\xrightarrow{T'} v_{i_2} \rightarrow v_{j+1}$
δ(v_{j+1}) ≤ d(v_{j+1})

Need to prove: \forall \text{path } \Gamma \rightarrow v_{j+1}, : |\Gamma| ≤ d(v_{j+1})

Case (1) |\Gamma| = 0. Clearly 0 ≤ d(v_{j+1}).

Case (2) |\Gamma| > 0. Let v_i denote the predecessor of v_{j+1} along the path \Gamma.

Γ = Γ′ o (v_i, v_{j+1}):

Topo. sort: i < j+1.

Ind. hyp.: |Γ′| ≤ d(v_i)

Hence:

|Γ| = 1 + |Γ′| ≤ 1 + d(v_i) ≤ d(v_{j+1})

Why?
Q: how can we actually find a longest path?

longest-path just tells us its length. Can we find a longest path?