Chapter 4: Directed Graphs

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Book Homepage:
http://www.eng.tau.ac.il/~guy/Even-Medina
In the following definition we consider a directed acyclic graph $G = (V, E)$ with a single sink called the root.

**Definition**

A DAG $G = (V, E)$ is a **rooted tree** if it satisfies the following conditions:

1. There is a single sink in $G$.
2. For every vertex in $V$ that is not a sink, the out-degree equals one.

The single sink in rooted tree $G$ is called the root, and we denote the root of $G$ by $r(G)$. 
Rooted Trees

acyclic directed graph $G = (V, E)$ s.t.

1) Single Sink
2) $\forall v: \text{deg}_{out}(v) \leq 1$

not rooted trees

rooted tree
Definition

A DAG $G = (V, E)$ is a rooted tree if it satisfies the following conditions:

1. There is a single sink in $G$.
2. For every vertex in $V$ that is not a sink, the out-degree equals one.

Theorem

*In a rooted tree there is a unique path from every vertex to the root.*
\( G = (V, E) \) rooted tree \( \Rightarrow \forall v \in V \exists ! \text{path } v \rightarrow^* \text{root} \)

proof by ind. on \( |V| \).

base: \( |V| = 1 \), trivial.

hyp: holds if \( |V| = n \).

step: prove for \( |V| = n+1 \).

\( G \text{ DAG } \Rightarrow \exists \text{ source } v \)

consider \( G' = (V', E') \) where \( \begin{cases} V' = V \setminus \{v\} \\ E' = E \setminus Ev \end{cases} \)

\( G' \) is a rooted tree: \( \text{deg}_{G'}(u) \) is unchanged

ind. hyp on \( G' : \forall u \in V' \exists ! \text{path } u \rightarrow^* \text{root} \)

what about \( v ? \) \( \text{deg}_{G'}(v) = 1 \) \( \Rightarrow \exists ! u : (v,u) \in E \)

\begin{center}
\begin{tikzpicture}
    \node (v) at (0,0) {$v$};
    \node (u) at (2,0) {$u$};
    \node (root) at (4,0) {root};
    \draw[->] (v) -- (u);
    \draw[->] (u) -- (root);
    \node at (1,-0.5) {uniq. path};
    \node at (-0.5,0) {\text{only edge}};
\end{tikzpicture}
\end{center}
2nd proof: 1) \exists \text{ path to root}

2) unique path to root

\exists \text{ path to root: }

pick \( v \in V \). build path recursively as follows:

\[ V_0 \leftarrow v \]

if \( v \) \( \text{Sink} \) \( \text{stop} \).

if \( v \neq \text{sink} \), \( \exists u : (v_i, u) \in E \).

set \( v_{i+1} \leftarrow u \).

since \( |\text{path}| < \infty \), alg. must terminate.

sink is unique, path reaches the root.
2) unique path to root

if \exists 2 paths: v \rightarrow root

\[ \text{paths diverge} \]
\[ \Rightarrow \text{deg}_{\text{out}}(u) \geq 2 \]

\[ \Rightarrow \text{contra. to } G \text{ is a rooted tree.} \]
Figure: A decomposition of a rooted tree $G$ into two rooted trees $G_1$ and $G_2$. 
Terminology

- each the rooted tree $G_i = (V_i, E_i)$ is called a tree hanging from $r(G)$.
- **Leaf**: a source node.
- **interior vertex**: a vertex that is not a leaf.
- **parent**: if $u \rightarrow v$, then $v$ is the parent of $u$.
- Typically maximum in-degree = 2.
The rooted trees hanging from \( r(G) \) are **ordered**. Important in parse trees.

Arcs are oriented from the leaves towards the root. Useful for modeling circuits:
- leaves = inputs
- root = output of the circuit.

\[ a \div (b + c) \]

\[ (X \text{ AND } Y) \text{ OR } Z \]