One-to-one and Onto Functions

Definition

Let $f : A \to B$ denote a function from A to B.

1. The function f is one-to-one if $a \neq a'$ implies that $f(a) \neq f(a')$.
2. The function f is onto if, for every $b \in B$, there exists an $a \in A$ such that $f(a) = b$.
3. The function f is a bijection if it is both onto and one-to-one.

- A one-to-one function is sometimes called an injective function (or an injection).
- A function that is onto is sometimes called a surjection.

Restrictions of one-to-one functions

Lemma

Every restriction of a one-to-one function is one-to-one.

Comparing size with one-to-one functions

Lemma (2.5)

Let A and B denote two finite sets. If there exists a one-to-one function $f : A \to B$, then $|A| \leq |B|$.

Pigeonhole Principle

- By Lemma 2.5: If there exists a one-to-one function $f : A \to B$, then $|A| \leq |B|$.
- The contrapositive form of Lemma 2.5: if $|A| > |B|$, then every function $f : A \to B$ is not one-to-one.

We are now ready to formalize the Pigeonhole Principle, as follows.

The Pigeonhole Principle

Let $f : A \to \{1, \ldots, n\}$, and $|A| > n$, then f is not one-to-one, i.e., there are $a_1, a_2 \in A$; $a_1 \neq a_2$, such that $f(a_1) = f(a_2)$.