Approximation Algorithms for Capacitated Rectangle Stabbing

Note Title 5/24/2006

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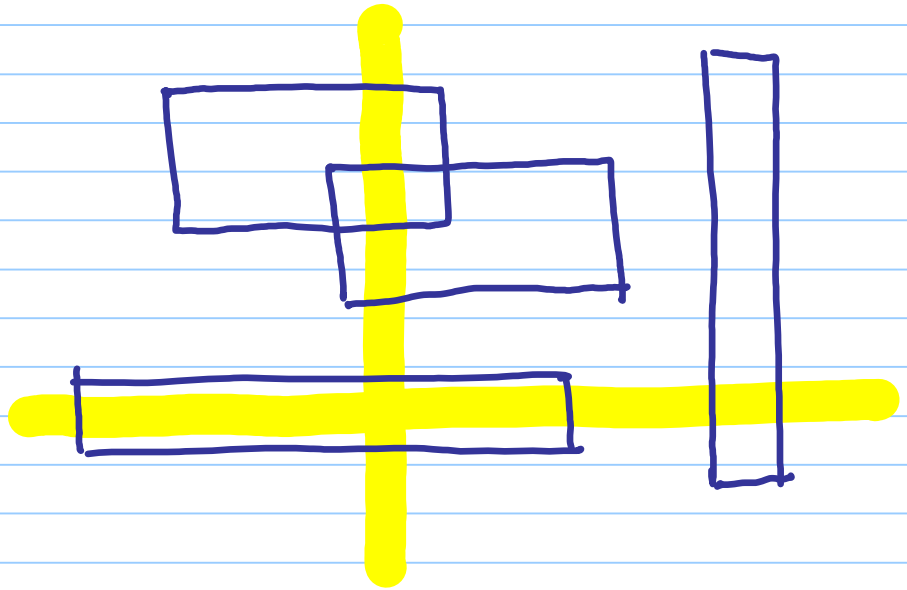
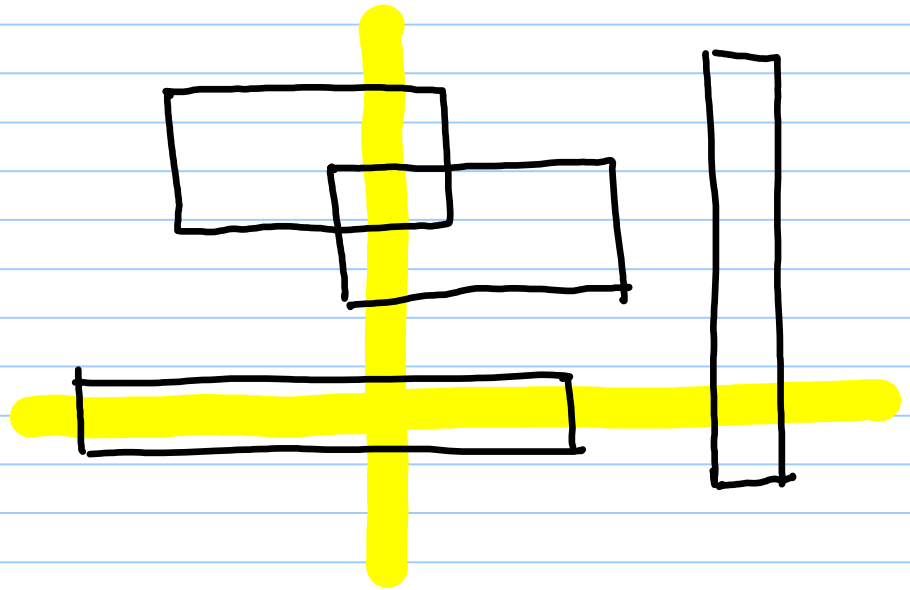


CIAC - 2006

Rectangle Stabbing (RS)



INPUT

* rectangles (axis parallel)
* horz/vert lines

GOAL

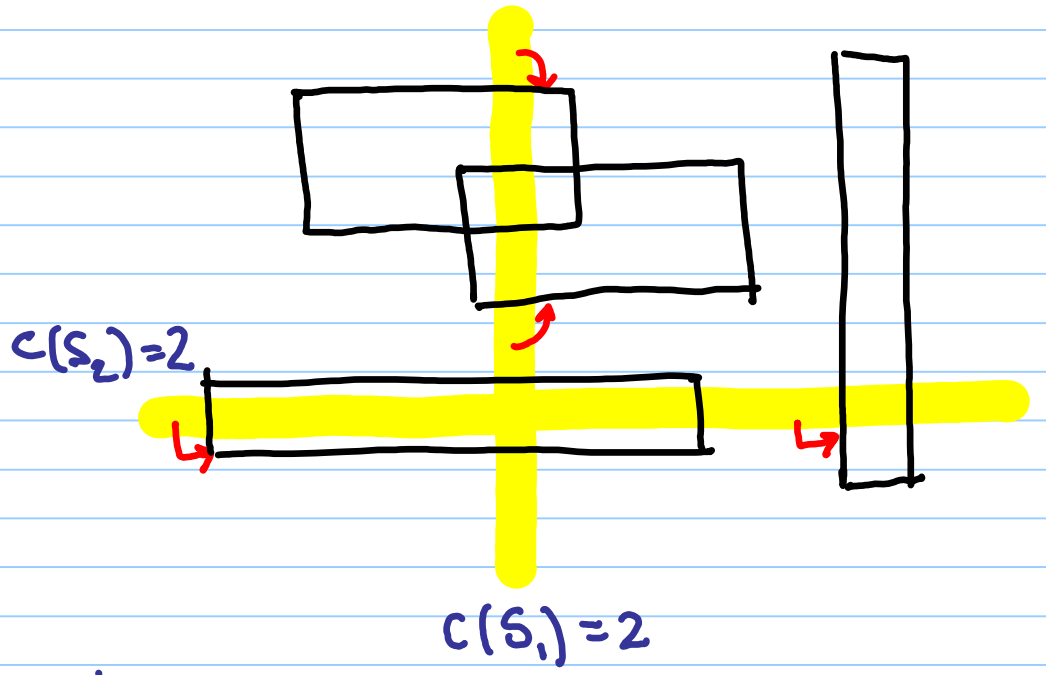
stab every rectangle with MIN #lines.

* one-dim: famous clique cover in interval graphs
* 2-dim: NP-C (Hassin & Megiddo)
* $d \geq 2$: d-apx algorithm (Gaur, Ibaraki, Krish.)

Capacitated - RS

Capacities $c(S)$ given to lines:

$c(S)$: max #rectangles assigned to line S.



2 versions: HARD & SOFT

HARD: capacities must be observed!

SOFT: may "buy" multiple copies of a line

$$\#\text{copies} = \left\lceil \frac{\#\text{rects assigned to } S}{c(S)} \right\rceil$$

GOAL: MIN #lines in capacitated cover.

Approx Algs for Capacitated Covering Problems

Covering Prob.		Cap. Covering Prob.	
Set Cover	LB: $\ln n$ UB: $\ln n + 1$	Cap Set cover	UB: $\lg n$ (W)
Vertex Cover	LB: ~1.36 UB: $2 - o(1)$	cap. vertex cover	UB: 2 (GHK, soft; CN, GHKKS, hard)

Q: do capacities make covering harder?

A: yes! bi-partite vertex cover (P vs. NP) (S)

Results

1) 1-dim cap-RS with soft caps : 2-approx.

2) D-dim cap-RS with soft caps : 6·D - approx.

3) D-dim cap-RS with hard caps : (16D, 2) - bicrit.

4) 1-dim cap-RS with hard caps : 8-approx

5) NP-hardness : i) 2-dim weighted cap-RS (Set-Cover hard)

(ii) D-dim RS (if D is part of the input : $\Omega(\log D)$-approx is NPC).

One-dimensional Cap-RS

Rectangles $\longmapsto$ horz. intervals

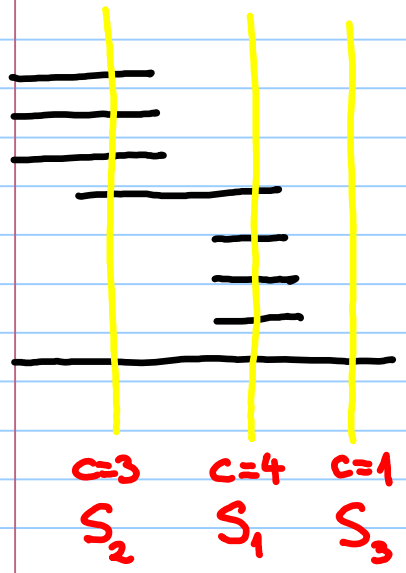
lines $\longmapsto$ vert. lines

without capacities:

clique cover in interval graphs (consecutive 1's prop. $O(n)$ time alg.)

We consider 2 cases: [redacted] & [redacted] capacities.

ALG : 1-Dim Cap-RS (Soft Caps)



* SORT lines in descending caps order.

* decisive cover - for $L \subseteq$ lines, assign every interval u to the [redacted] line S that intersects u.

Claims

1) OPT decisive cover $\leq 2 \cdot$ OPT
2) Dynamic Programming finds OPT decisive.

$\Rightarrow$ 2-approx alg.

Remark: extends to weights.

LP-Relaxation [CN]

$x(S) \in [0,1]$ - "portion" of line S in frac. cover.

$y(S,u) \in [0, x(S)]$ - coverage of rectangle u by S.

$$\text{MIN} \quad \sum_S x(S)$$

s.t.

$$\sum_{S:\, S \cap u \neq \emptyset} y(S,u) \geq 1 \qquad \forall \text{rect } u$$

$$\sum_u y(S,u) \leq c(S) \cdot x(S) \qquad \forall \text{line } S$$

int. gap unbounded without this constr.

$$y(S,u) \leq x(S) \qquad \forall \text{rect } u, \forall \text{line } S$$

$$x(s),\ y(S,u) \geq 0 \qquad \forall \text{rect } u, \forall \text{line } S$$

Integrality gap of LP-relaxation

$$\text{MIN} \quad \sum_S x(S)$$

s.t.

$$\sum_{S: u \in S \neq \emptyset} y(S,u) \geq 1 \qquad \forall \text{rect } u$$

$$\sum_u y(S,u) \leq c(S) \cdot x(S) \qquad \forall \text{line } S$$

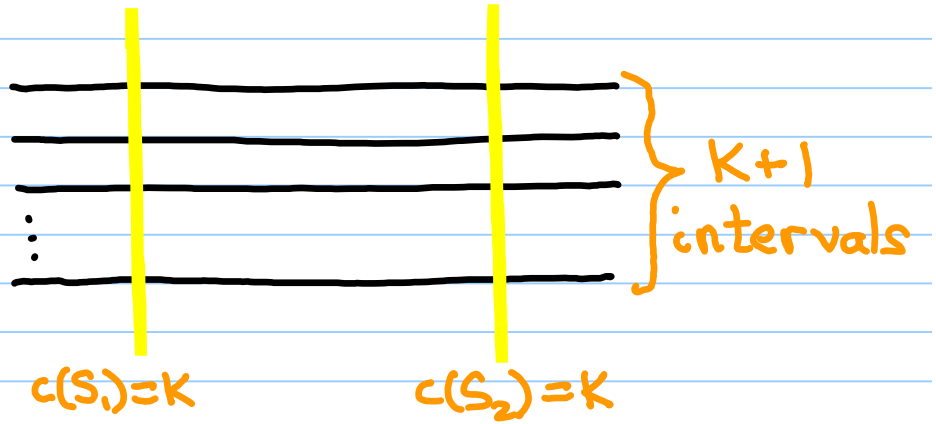
$$y(S,u) \leq x(S) \qquad \forall \text{rect } u, \forall \text{line } S$$

$$x(S),\ y(S,u) \geq 0 \qquad \forall \text{rect } u, \forall \text{line } S$$

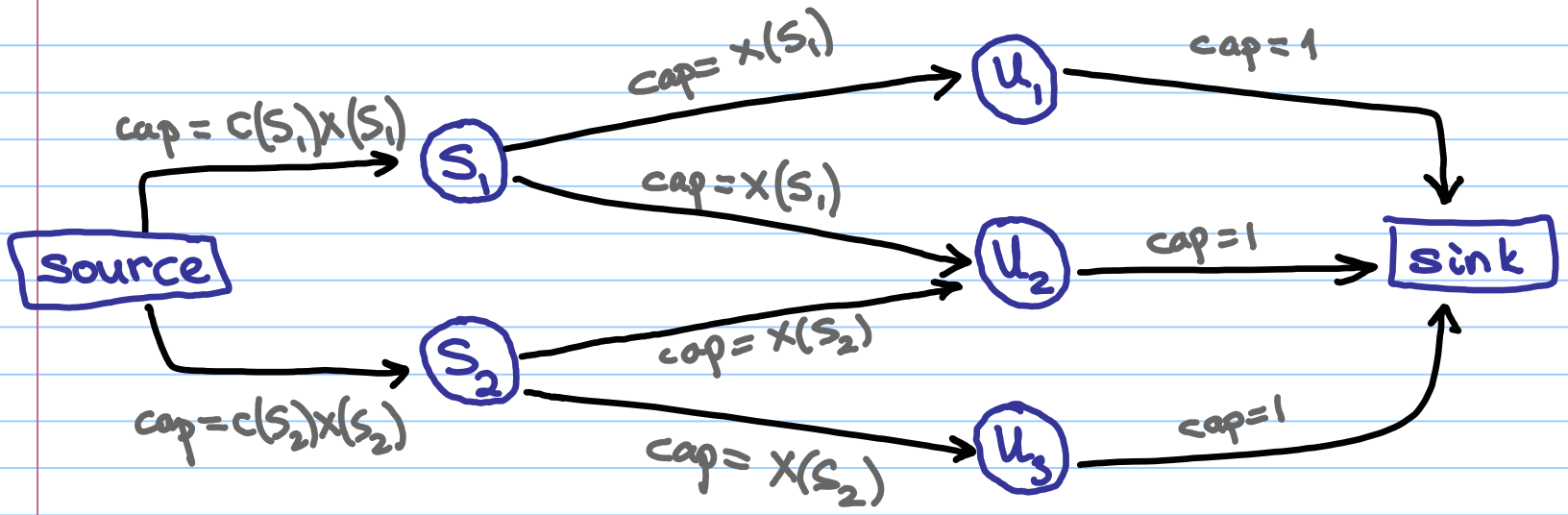
Integrality gap $\geq 2 - o(1)$

$$OPT(INT) = 2$$

$$OPT(FRAC) = 2 \times \frac{K+1}{2K} = 1 + \frac{1}{K}$$



Network flow formulation [CN]



For every $\{x(s_i)\}$, can compute a max flow, where

Flow on $(s_i, u_j) = y(s_i, u_j)$

COROLLARIES:

"integrality"

1) if $x(s_i) \in \{0,1\}$, then $y(s_i, u_j) \in \{0,1\}$.

"domination"

2) $\forall (x,y)\ \exists (x,y')$: $\sum_u y'(s,u) \geq \sum_u y(s,u)$ & $\sum_s y'(s,u) \geq \sum_s y(s,u)$

Plan

Round (x^*, y^*) – an optimal LP-solution

$\longrightarrow$ $O(D)$ approx alg for D-dim soft-cap-RS

$\longrightarrow$ $(O(D), 2)$-bicriteria approx for D-dim hard-cap-RS

$\longrightarrow$ starting point for 8-approx of 1-dim hard-cap-RS

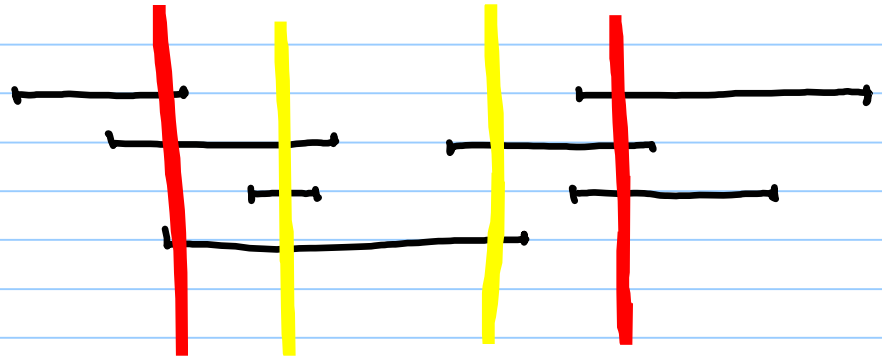
IDEA: distinguish between

$$H = \{ \text{lines } S : x^*(S) \geq \varepsilon \}$$

$$L = \{ \text{lines } S : x^*(S) < \varepsilon \}$$

Focus on 1-D case ....

Rounding an LP solution (x^*, y^*)



* $H = \{\text{lines } S : x^*(S) \geq \varepsilon\}$

* round $\forall S \in H$: $\begin{cases} x(S) = 1 \\ y(S,u) = y^*(S,u) \end{cases}$,

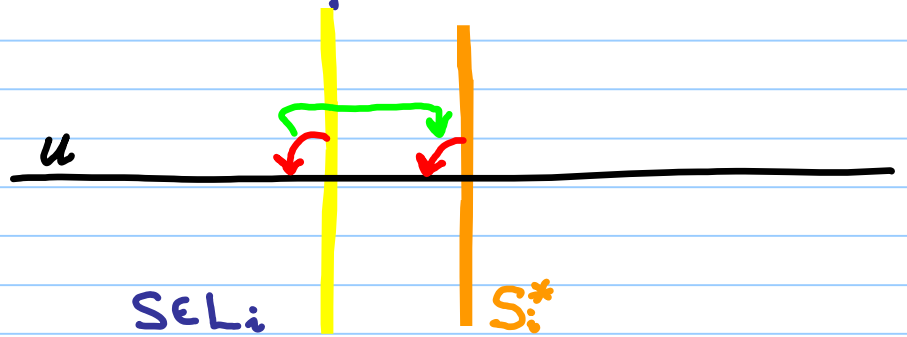
* simple... so assume $H = \emptyset$.

rounding LP-solution (x^*, y^*)

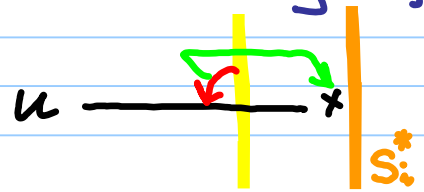
* assume $x^*(S) < \varepsilon$, $\forall$ line S
* partition lines into blocks L_i such that $x^*(L_i) \in [\varepsilon, 2\varepsilon)$

* pick line $S_i^* \in L_i$ with max capacity.
* "reroute" coverage in L_i to S_i^* (when possible)

if $u \cap S_i^* \neq \phi$: $y(S_i^*, u) \triangleq \sum_{S \in L_i} y^*(S, u)$



* coverage "lost" only if coverage$(u, L_i) > 0$ but $u \cap S_i^* = \phi$



Loss $< 2\varepsilon$ at each end $< 4\varepsilon$

rounding LP-solution (x^*, y^*)

$$x(S_i^*) = x^*(L_i), \quad y(S_i^*, u) = \sum_{S \in L_i} y^*(S, u) \quad \text{if } u \cap S_i^* \neq \emptyset$$

properties of (x,y)

1) (x,y) obeys capacity constraints of the LP.

2) $\sum_{S_i^*} x(S_i^*) \leq OPT^*$, $\forall i: \; x(S_i^*) \geq \varepsilon$

3) $\forall$ interval u: $\sum_S y(S,u) \geq 1 - 4\varepsilon$

$\varepsilon = 1/6$ ↓

$\geq \frac{1}{3}$ ← "$\frac{1}{3}$-cover"

Scale

$$x'(S) = \lceil 3 \cdot x(S) \rceil, \quad y'(S,u) = 3 \cdot y(S,u)$$

$\Rightarrow$ (x', y) is a "soft" cover with

$$\sum_S x'(S) \leq 6 \cdot OPT^*$$

← 6D-approx ratio for D-dim soft-RS

[side remark: INT GAP ≤ 6 for unweighted.]

If $\varepsilon = 1/8$, then $\forall$ interval is $\frac{1}{2}$-covered by (x,y).

$\Rightarrow$ at most two copies of each lines are needed.

$\Rightarrow$ bicriteria $(16D, 2)$-approx. alg for Hard-cap-RS.

Challenge: get rid of bicriteria ....

IDEA: * $|H| + \#\text{blocks of } L \leq \frac{1}{\varepsilon} \cdot OPT^*$

* start with partial cover $H \cup \{S_1^*, \ldots, S_k^*\}$

* iteratively add a new line & charge a different line in $H \cup \{S_i^*\}_i$ for the new line.

coverage of $\forall$ interval is $> 1 - 4\varepsilon$

Summary

* Investigate a simple family of capacitated covering prob.

* $O(D)$-approx for Soft-cap-RS
  bicriteria $(O(D), 2)$-approx for hard-cap-RS
  $O(1)$-approx for Hard 1-dim cap-RS.

* Hardness $\Rightarrow$ if D is part of input, then approx-ratio is $\Omega(\lg D)$.

OPEN PROBLEMS:

* hardness of 1-dim cap-RS (soft/hard caps).

* $O(D)$-approx alg. for D-dim Hard-cap-RS.
* $O(D)$-approx alg. for [redacted] D-dim soft-cap-RS.