$O(k^{1/2+\varepsilon})$-approximation algorithm for the directed Steiner Network problem

(How to "cheaply" connect k source-sink pairs in a directed graph?)

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(based on SODA 2008)

Theme

Greedy Algorithm: accumulate "good" partial solutions until solution is obtained

Good partial solution: $\min\left(\frac{\text{cost}}{\text{benefit}}\right)$ a.k.a. min-density

Study a setting where finding a min-density partial solution is NP-hard:

- Consider restricted partial solutions

  need to bound penalty of restriction

- approximate min-density restricted partial solution.

  find good approx alg

Problem: directed Steiner network (= cheaply connect source-sink pairs)

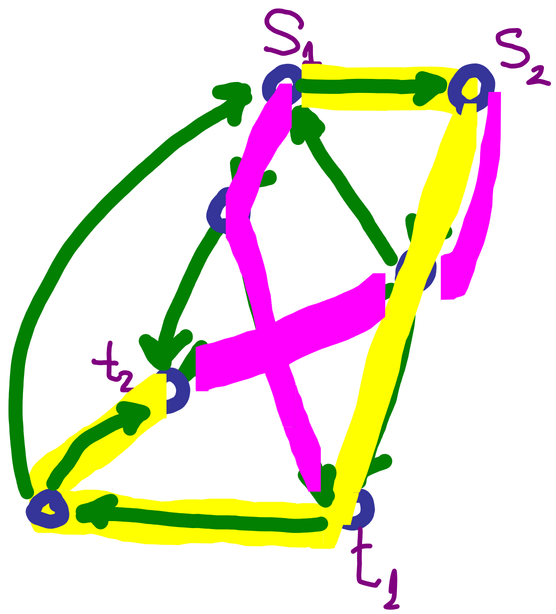
Input: directed graph $G = (V, E)$

k source-sink pairs $\{(s_i, t_i)\}_{i=1}^{k}$

arc weights $w : E \to \mathbb{R}$

Output: A subset of arcs $F \subseteq E$ such that

$\forall 1 \le i \le k$: $\exists$ path $s_i \rightsquigarrow t_i$ consisting only of arcs in F.



Goal: Minimize $w(F) = \sum_{e \in F} w(e)$

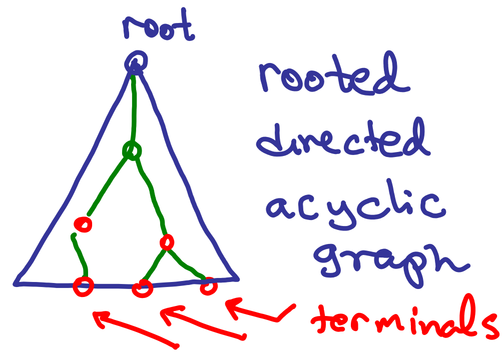
history: directed Steiner network

THM [Dodis-Khanna 99]: no $O(2^{\lg^{1-\varepsilon} n})$-apx $(\forall \varepsilon > 0)$

if $NP \not\subseteq DTIME(n^{\text{polylog}(n)})$

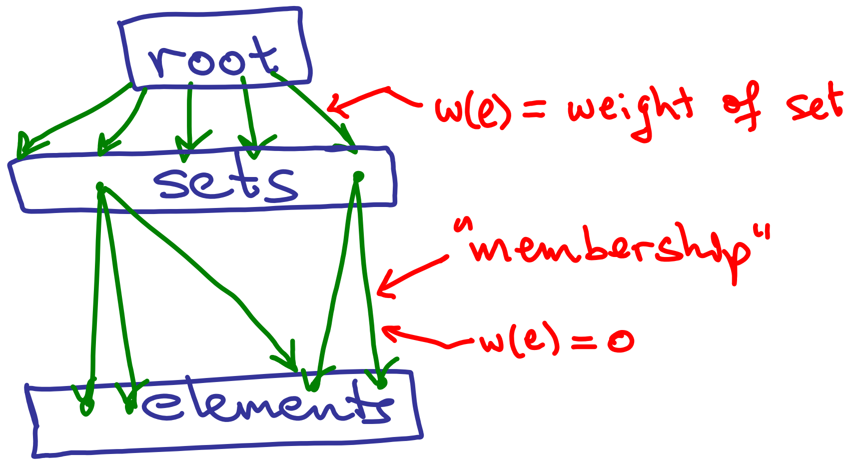
A weaker hardness result:

- Generalizes: directed Steiner tree



(find a min weight rooted tree in a DAG that spans a set of terminals)

which generalizes Set-Cover by a reduction to a height 2 tree:



$\Rightarrow$ NP-C even to approx with $O(\lg n)$-ratio.

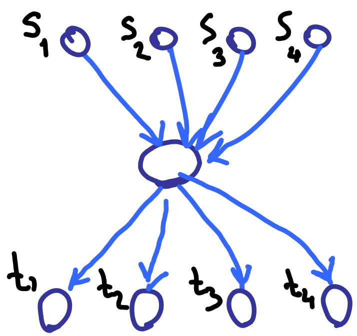
history: directed Steiner network

approximation algorithm: $\tilde{O}(k^{2/3})$-ratio by

Charikar, Chekuri, Cheung, Dai, Goel, Guha, Li [99].

based on a "greedy" algorithm that accumulates bunches of optimal density until all pairs are connected.

bunch



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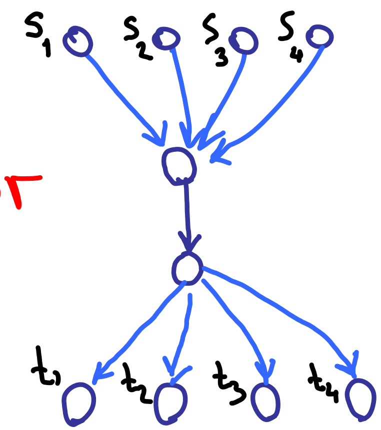
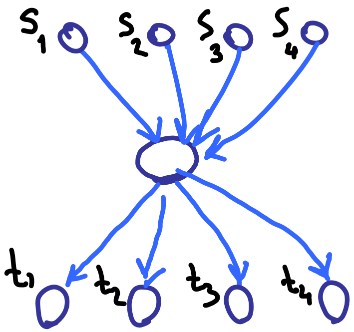


$$\text{density(bunch)} \triangleq \frac{\sum_{e \in \text{bunch}} w(e)}{\#\text{ terminal pairs in the bunch}}$$

bunch



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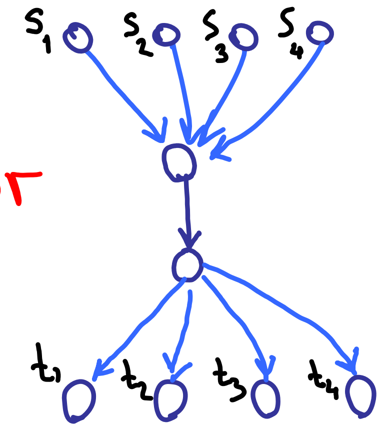


$$\text{density(bunch)} \triangleq \frac{\sum_{e \in \text{bunch}} w(e)}{\#\text{ terminal pairs in the bunch}}$$

Lemma: [CCC+99]

1. easy to find bunch of min density.

not easy!

2. $\text{MIN density(bunch)} \leq \tilde{O}(k^{2/3}) \cdot \dfrac{OPT_k}{k}$

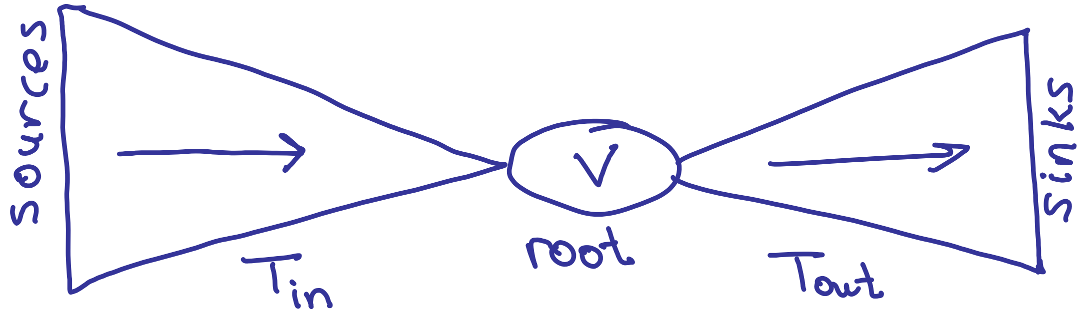
weight of optimum directed Steiner network

Open: [CCC+99]

1. $\sup \dfrac{\text{Min density(bunch)}}{OPT_k / k} = \, ?$

2. improve approx ratio ...

Junction Trees - Definitions



<u>remarks:</u>

1) T_{in} and T_{out} may intersect

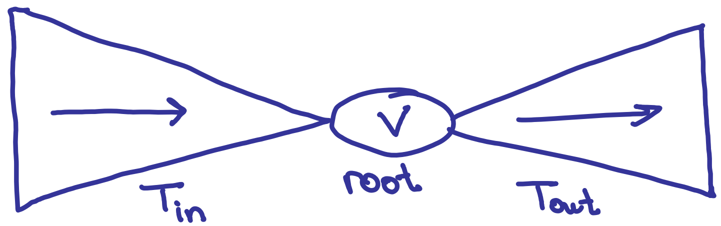
2) union of directed paths that traverse root.

Given a junction tree JT, define

$$D(JT) \triangleq \#\text{ source-sink pairs connected by } JT$$

$$\text{density}(JT) \triangleq \frac{w(JT)}{D(JT)}$$

Junction Trees



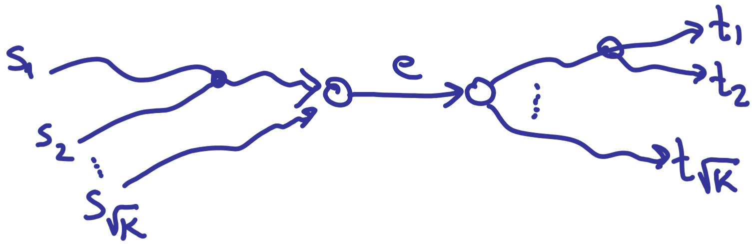
$$\text{density}(JT) \triangleq \frac{w(JT)}{D(JT)}$$

Lemma: $\exists\, JT$:

$$\text{density}(JT) \leq \sqrt{k} \cdot \frac{OPT_k}{k}$$

Proof: Fix $OPT_k = p_1 \cup p_2 \cup \cdots p_k$ where $(s_i \overset{p_i}{\rightsquigarrow} t_i)$. Two cases:

(I) $\exists$ arc e that is contained in $\sqrt{k}$ paths.



$$\text{density}(JT_e) \leq \frac{OPT_k}{\sqrt{k}}$$

(II) otherwise:
$$\min_i w(p_i) \leq \frac{\sum w(p_i)}{k} \leq \frac{\sqrt{k} \cdot OPT_k}{k}$$

$\Rightarrow$ so lightest source-sink path is good.

Junction Trees

Lemma: $\exists\, JT$: density$(JT) \leq \sqrt{k} \cdot \frac{OPT_k}{k}$

Idea for algorithm: accumulate junction trees of (approx) min density until all source-sink pairs are connected.

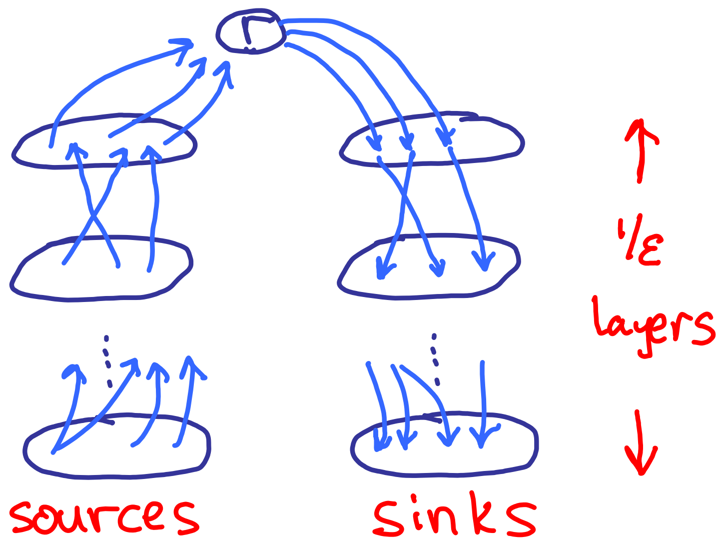
Problem: as opposed to bunches, junction trees of min density are hard to find!

1) Even if source-sink pairs in JT are known: as hard as directed steiner tree.

2) How do we decide which pairs to connect? (solve separatly for sources & sinks, but what if don't match?)

Simplifying assumption - obtained by $\begin{cases}\text{transitive closure} \\ + \\ \text{Layering}\end{cases}$

* root of JT known
* graph = 2 layered DAGs
* # layers constant $(= \frac{1}{\varepsilon})$



Lemma (Zelikowsky):

Assumption incurs a penalty of

$O(k^{\varepsilon})$ in approx ratio.

Reduction: min density JT $\leq$ min density General Connectivity

(min density) general connectivity [Alon, Awerbuch, Azar, Buchbinder, Naor 06]

considered an on-line setting

Input: undirected graph $G=(V,E)$

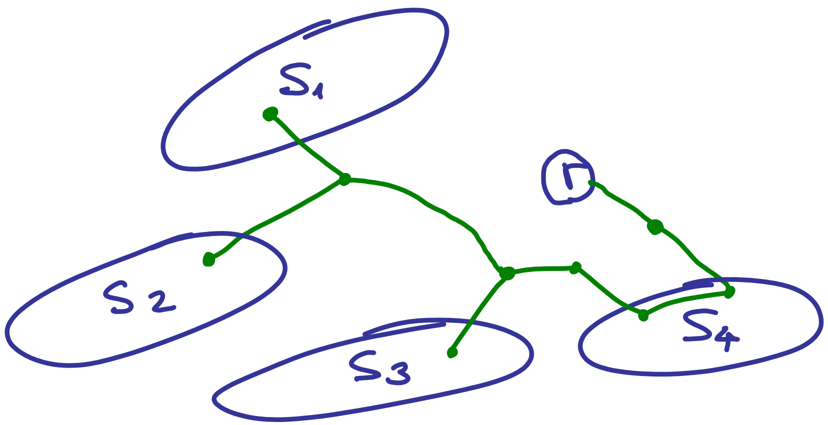
edge weights $w: E \to \mathbb{R}$

root r

terminal sets $S_1, \dots, S_k \subseteq V$

Output: r-rooted tree $T=(U,F)$ such that $\forall i: S_i \cap U \neq \emptyset$

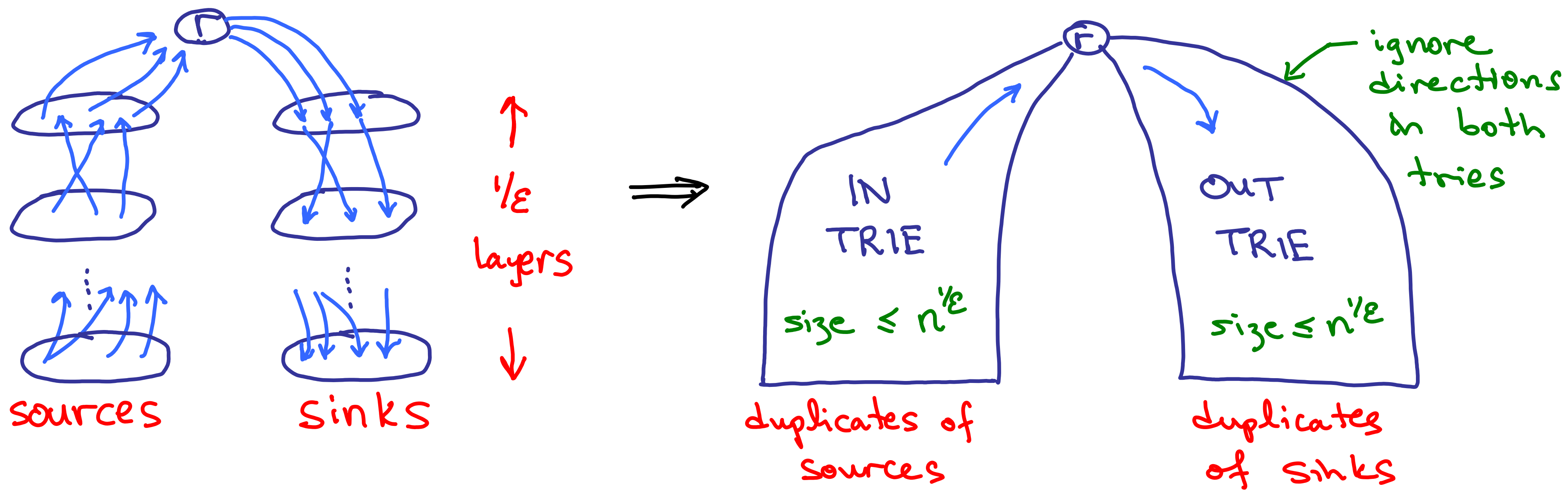
Goal: $\min \quad w(F) \triangleq \sum_{e \in F} w(e)$.



min-density version:

$$\min \frac{w(F)}{\#\{i : S_i \cap U\}}$$

Reduction: min density JT $\leqslant$ min density General Connectivity



IN-TRIE = enumerate all source $\rightsquigarrow$ root paths

$$S_i = \{ \text{leaves in in-trie : leaf corresponds to } s_i \}$$

$$T_i = \{ \text{leaves in out-trie : leaf corresponds to } t_i \}$$

Trivial: rooted trees that connect $S_i - T_i$ $\xleftrightarrow{\text{cost preserving correspondence}}$ junction trees that connect $s_i - t_i$

Q: how to solve min density general connectivity?

again, 2 difficulties:

- which pairs $S_i - T_i$ should we connect?
- How to connect? (group Steiner problem over a tree with constant height)

LP-relaxation

$$\min \sum_{e \in E} w(e) \cdot x_e$$

s.t.

$$\sum_{i=1}^{K} y_i = 1$$

$$\sum_{e \in \delta(U)} x_e \geq y_i$$

$\forall 1 \leq i \leq K$
$\forall U \subseteq V$ that disconnects root from S_i or T_i

$$1 \geq x_e, y_i \geq 0$$

$K \cdot y_i$ - indicates whether $S_i - T_i$ connected

x_e - is e selected to the solution?

Lemma: min density general connec. $\geq \sum_{e\in E} w(e)\, x_e^*$ $\leftarrow$ optimal solution of LP

Proof: given $F \subseteq E$, let $K(F) \triangleq$ # connected $S_i - T_i$ pairs.

$$x_e \triangleq \begin{cases} \frac{1}{K(F)} & \text{if } e\in F \\ 0 & \text{otherwise} \end{cases} \qquad y_i \triangleq \begin{cases} \frac{1}{K(F)} & \text{if } S_i\text{-}T_i \text{ connected by } F \\ 0 & \text{otherwise} \end{cases}$$

$$\text{density}(F) = \frac{w(F)}{K(F)} = \sum_{e\in E} w(e)\, x_e$$

What do we do with optimal solution x^*, y^*?

How does an optimal solution (x^*, y^*) help in finding an (approx) min density gen. connectivity solution?

Goal: find $F \subseteq E$: $\dfrac{w(F)}{\#\{i : S_i \overset{F}{-} T_i\}} \leq \sum_E w(e)\cdot x_e^* \cdot \text{poly-log}(K)$

Bucket & Scale

I_j is a "bucket"

def $I_j \triangleq \{ i \mid y_i^* \in (2^{-j}, 2 \times 2^{-j}] \}$

claim $\sum_{1 \le j \le \log_2(2k)} \left(\sum_{i \in I_j} y_i^* \right) \ge \frac{1}{2}$

claim $\exists\, 1 \le j^* \le \log_2(2k) : \sum_{i \in I_{j^*}} y_i^* \ge \frac{1}{2 \lg 2k}$

LP- relaxation

$$\min \sum_{e \in E} w(e) \cdot x_e$$

s.t.

$$\sum_{i=1}^{k} y_i = 1$$

$$\sum_{e \in \delta(U)} x_e \ge y_i \qquad \forall\, 1 \le i \le k,\ \forall\, U \subseteq V \text{ that disconnects root from } S_i \text{ or } T_i$$

$$1 \ge x_e, y_i \ge 0$$

Bucket I_{j^*} defines the set of $S_i - T_i$ pairs to connect!

But: we need to connect these pairs using $F \subseteq E$ such that $\frac{w(F)}{|I_{j^*}|} \le \sum_E w(e)\, x(e) \cdot O(\lg k)$ ... how?

use LP relaxation of group Steiner ...

$$I_{j^*} = \{ i \mid y_i \in (2^{-j^*}, 2 \times 2^{-j^*}] \} \quad \& \quad \sum_{i \in I_{j^*}} y_i^* \geq \frac{1}{2 \lg 2K}$$

LP-relaxation for gen. conn.

$$\min \sum_{e \in E} w(e) \cdot x_e$$

s.t.

$$\sum_{i=1}^{K} y_i = 1$$

$$\sum_{e \in \delta(U)} x_e \geq y_i \qquad \forall 1 \leq i \leq K \;\; \forall U \subseteq V \text{ that disconnects root from } S_i \text{ or } T_i$$

$$1 \geq x_e, y_i \geq 0 \qquad \forall i, \forall e$$

LP(I_{j^*})

$$\min \sum_{e \in E} w(e) \cdot x_e$$

s.t.

$$\sum_{e \in \delta(U)} x_e \geq 1 \qquad \forall i \in I_{j^*} \;\; \forall U \subseteq V \text{ that disconnects root from } S_i \text{ or } T_i$$

$$1 \geq x_e \geq 0 \qquad \forall e$$

Claim: $2^{j^*} \cdot x^*$ is feasible for LP(I_{j^*}) ← scale by 2^{j^*}

$$\text{density}(2^{j^*} \cdot x^*) = \frac{2^{j^*} \cdot \sum_E w(e) x_e}{|I_{j^*}|} \leq \frac{2^{j^*} \cdot \sum_E w(e) x_e}{\frac{1}{2} \cdot 2^{j^*} \cdot \sum_{i \in I_{j^*}} y_i^*} \leq \sum_E w(e) x_e \cdot (4 \cdot \lg 2K)$$

Recap: we have a fractional solution $2^{j^*} x^*$ that solves $LP(I_{j^*})$ with good density.

Goal: Round $2^{j^*} x^*$ to find an integral $F \subseteq E$ with similar density.

How: Apply rounding procedure for group steiner [Garg, Konjevod, Ravi - 2000]

$\Rightarrow$ Find $F \subseteq E$ that connects half of I_{j^*} with density $\leq \sum_E w(e) x_e^* \cdot O(\lg k)$

Summary (for directed Steiner network)

1) Apply greedy algorithm by accumulation approx min density junction trees $\leftarrow$ $\sqrt{K}$ penalty

2) Find approx min-density junction tree

   i) reduction to gen. conn. over an undirected tree of height $\frac{1}{\varepsilon}$ $\longleftarrow$ $O(K^{\varepsilon})$ penalty

   ii) approx by solving LP + rounding à la [GKR00]

   $O(\lg K)$ penalty

$\Longrightarrow$ approx ratio : $O\left(K^{\frac{1}{2}+\varepsilon}\right)$

Almost tight: $\sup \dfrac{\text{min-density(JT)}}{OPT_K / K} \geq \Omega(\sqrt{K})$

Merci Beaucoup!

Bon Voyage!