Linear Programming :

A view from the Set-Cover perspective

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Systems of equations $Ax = b$

solution : an (affine) subspace of $\mathbb{R}^n$

$z + \{x : Ax = b\}$ where $Az = b$

algorithm: Gauss elimination

What about inequalities ?

$$\begin{cases} Ax = b \\ x \geq 0 \end{cases} \qquad \begin{cases} Ax \leq b \end{cases} \qquad \begin{cases} x_1 + 2x_2 + 3x_3 \leq 4 \\ x_1 + x_2 + x_3 = 1 \\ x_1 \geq 0 \\ x_2 \geq 0 \\ x_3 \geq 0 \end{cases}$$

Question #1 Find an efficient algorithm for the following problem:

Input: a set Π of constraints ((in)equalities over variables $x_1, \dots, x_n \in \mathbb{R}$)

[Each constraint is of the form:

$$a_{i1} \cdot x_1 + a_{i2} x_2 + \cdots + a_{in} x_n \overset{=}{\underset{\geq}{\leq}} b_i$$]

Output: "not feasible" OR $\vec{x} \in \mathbb{R}^n$ that satisfies all the constraints

Question #2: Find an efficient algorithm for the following problem:

Input: $m \times n$ matrix $A \in M_{m \times n}$

vector $b \in \mathbb{R}^m$

vector $c \in \mathbb{R}^n$

Goal: find $x \in \mathbb{R}^n$ that minimizes $c^t \cdot x \triangleq \sum_j c_j x_j$ subject to $A \cdot x = b$, $x \geq 0$.

Output: "not feasible" or $x^* = \operatorname{argmin}\{c^t x \mid Ax = b,\ x \geq 0\}$.

Both questions are about linear programming !

- Q1 : Is a polyhedron empty ?
- Q2 : Linear programming in standard form.

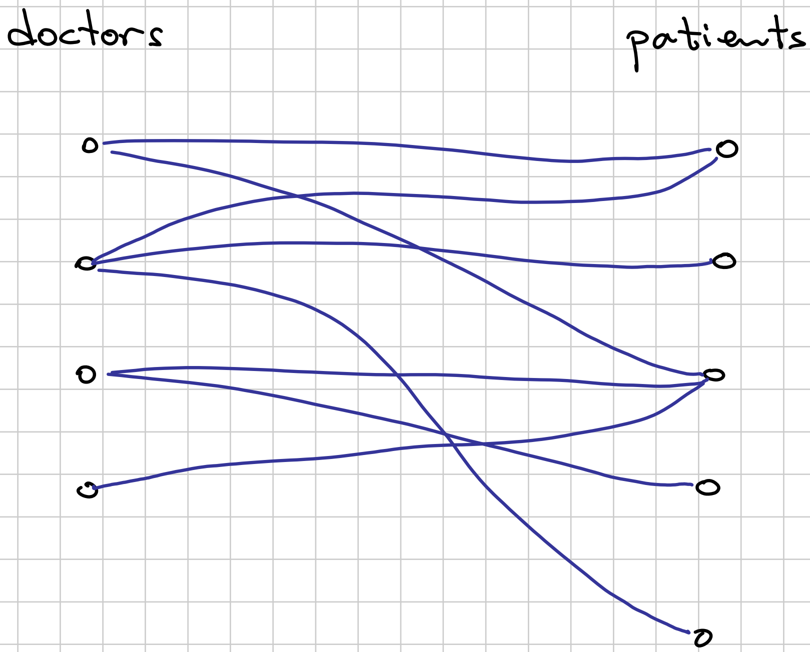
Lots of : structure
& algorithms ...

Combinatorial Optimization Problems

find "best" solution for a combinatorial problem (as opposed to a "numerical problem").

Examples : Set Cover
Vertex Cover
Minimum weight matching
Maximum flow

Set Cover - Example



Rules:

Every doctor can serve a subset of the patients.

Goal:

Choose fewest # doctors, so that each patient is served.

Definition: Set-Cover

Input: a set of elements $[1..n]$ patients

a family of subsets: $S_1, \dots, S_m \subseteq [1..n]$ doctors

$\{S_{i_1}, S_{i_2}, \dots, S_{i_k}\}$ is a Set Cover if

$$S_{i_1} \cup S_{i_2} \cup \dots \cup S_{i_k} = [1..n]$$

every patient served by a doctor

Goal: find a set cover of minimum size.

History : Set Cover

1) Lots of applications & special cases.

2) Among the first problems proven to be NP-Complete [Karp]

3) Greedy algorithm gives $H_n \approx \ln n$ approximation.

4) NP-Complete to find "better" approximations [Feige, RS]

What does linear programming have to say?

Set Cover : the greedy algorithm

Input: $S_1, \dots, S_m \subseteq [1..n]$

goal: find smallest set $\{S_{i_1}, \dots, S_{i_k}\}$ s.t. $S_{i_1} \cup \dots \cup S_{i_k} = [1..n]$

method: (Greedy)

$C \leftarrow \emptyset$

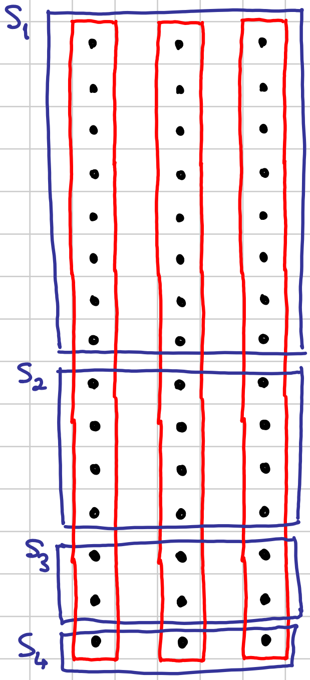
while $C \neq [1..n]$ do

$S \leftarrow \operatorname{argmax}\{ |S_i \setminus C| : 1 \leq i \leq m \}$

$C \leftarrow C \cup S$ (add S to the cover)

Question: How many sets in the cover (compared to an optimal solution)?

Greedy - Example



Greedy picks:

$$\{S_1, S_2, S_3, S_4\}$$

But $|OPT| = 3$

Set Cover: the greedy algorithm (CONT)

Theorem [Johnson]

greedy returns an (H_n)-approx.

$$H_n = 1+2+\dots+n \leq 1+\ln n$$

$$\left(|Greedy| \leq (1+\ln n)\cdot OPT \right)$$

An Integer Programming formulation for Set Cover

For each set S define a variable $x_S \in \{0,1\}$

$$\min \sum_S x_S$$
$$\text{s.t.} \quad \sum_{\{S : e \in S\}} x_S \geq 1 \quad \forall \text{element } e \in [1..n]$$

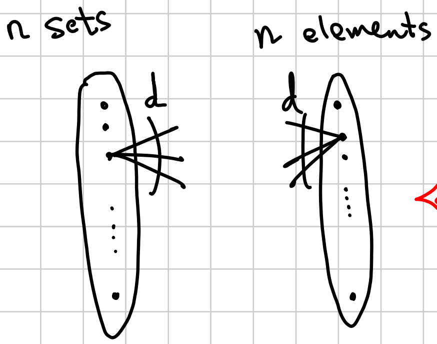
Now relax integrality constraint $x_S \in \{0,1\}$ and use fractional constraint $0 \leq x_S \leq 1$:

$$\min \sum_S x_S$$
$$\text{s.t.} \quad \sum_{\{S : e \in S\}} x_S \geq 1 \quad \forall \text{element } e$$
$$0 \leq x_S \leq 1 \quad \forall \text{set } S$$

fractional relaxation for Set-Cover

Example of fractional Set Cover

$$\min \sum_S x_S$$
$$\text{s.t.} \quad \sum_{\{S : e \in S\}} x_S \geq 1 \quad \forall \text{element } e$$
$$0 \leq x_S \leq 1 \quad \forall \text{set } S$$



d-regular bipartite graph (superposition of d matchings)

suppose $\forall$ set S : $|S| = d$

$\forall$ element e : $|\{S : e \in S\}| = d$

$\Rightarrow$ $x_S = \frac{1}{d}$ is a feasible solution.

$\sum x_S = \frac{n}{d}$ is optimal (exercise)

What can we do with (optimal) solution $\{x^*_S\}_S$ of LP?

$$\min \sum_S x_S$$
$$\text{s.t.} \sum_{\{S : e \in S\}} x_S \geq 1 \quad \forall \text{ element } e$$
$$0 \leq x_S \leq 1 \quad \forall \text{ set } S$$

1) Lower bound: $\sum_S x^*_S \triangleq OPT_f \leq OPT$

2) If (almost) integral then $\{x^*_S\}$ is (almost) a set cover.

3) Round $\{x^*_S\}$ ...

First, let's use it to analyze Greedy.

Greedy - with charging

$C \leftarrow \emptyset$

while $C \neq [1..n]$ do

- $S \leftarrow \operatorname{argmax}\{\,|S_i \setminus C| : 1 \leq i \leq m\,\}$
- $\forall e \in S \setminus C : \; p(e) \leftarrow \frac{1}{|S \setminus C|}$ (charge covered elements)
- $C \leftarrow C \cup S$ (add S to the cover)

In each round $\sum_{e \in S \setminus C} p(e) = 1$ pays for S.

$\Rightarrow$ In the end: $\sum_e p(e) = |C|$.

Why does this help?

Primal LP (covering)

$$\min \sum_S x_S$$
$$\text{s.t.} \sum_{\{S: e\in S\}} x_S \geq 1 \quad \forall \text{element } e$$
$$0 \leq x_S \leq 1 \quad \forall \text{set } S$$

Dual LP (packing)

$$\max \sum_e y_e$$
$$\text{s.t.} \sum_{e\in S} y_e \leq 1 \quad \forall S$$
$$0 \leq y_e$$

Claim (weak duality): $\sum_S x_S \geq \sum_e y_e$

proof: $\sum_S 1\cdot x_S \geq \sum_S \left(\sum_{e\in S} y_e\right) x_S = \sum_e \left(\sum_{S: e\in S} x_S\right) y_e \geq \sum_e y_e$.

Hope: If $\{p_e\}$ is a dual solution, then

$$|\text{Greedy}| = \sum_e p_e \leq OPT_f \leq OPT \ldots$$

But, $\{p_e\}$ is NOT a dual solution.... (exercise)

Lovasz: $y_e = \frac{p_e}{1+\ln n}$ is dual solution !

and then $|\text{Greedy}| = \sum_e p_e = (1+\ln n)\sum_e y_e$

$$\leq (1+\ln n)\cdot OPT_f \leq (1+\ln n)\cdot OPT$$

$\Rightarrow$ Greedy is an $O(\lg n)$-approximation.

So, now left to prove feasibility of $\{y_e\}$.

Consider any set S

$S = \{e_1, e_2, \dots, e_\ell\}$.

let $T(e) \triangleq$ iteration in which e is covered

while $C \neq [1..n]$ do

$S \leftarrow \arg\max\{ |S_i \setminus C| : 1 \leq i \leq m \}$

$\forall e \in S \setminus C : \ p(e) \leftarrow \frac{1}{|S \setminus C|}$ (charge covered elements)

$C \leftarrow C \cup S$ (add S to the cover)

assume: $T(e_1) \leq T(e_2) \leq \cdots \leq T(e_\ell)$

$$\Rightarrow p(e_1) \leq p(e_2) \leq \cdots \leq p(e_\ell)$$

moreover, when e_i is covered $S \setminus C \supseteq \{e_i, e_{i+1}, \dots, e_\ell\}$.

$$\Rightarrow p(e_1) \leq \frac{1}{\ell}, \ p(e_2) \leq \frac{1}{\ell-1}, \ \dots, \ p(e_\ell) \leq 1.$$

$$\Rightarrow \forall S : \sum_{e \in S} p(e) \leq \sum_{i=1}^{n} \frac{1}{i} \triangleq H_n \leq 1 + \ln n.$$

$$\Rightarrow y_e = \frac{p(e)}{1+\ln n} \text{ is dual feasible.} \qquad \square$$

Recap

1) Set Cover Problem

2) Greedy Algorithm

3) Integer Programming Formulation.

4) Linear Programming Formulation.

5) Primal & Dual LP's.

6) Weak Duality

7) Analysis of approximation ratio of Greedy.

Problems

1) Give an example where:

$$|\text{Greedy}| \geq \Omega(\lg n)$$

and

$$|\text{OPT}| \leq \text{constant}.$$

Hint: extend the example from slides.

* Does this show that $\{p(e)\}$ is <u>not</u> a dual solution?

* Can you give an example with $\dfrac{|\text{Greedy}|}{|\text{OPT}|} = H_n$?

Problems

2) Prove that if a set-system has n sets &
n elements in which $\forall$set $|S| = d$ and
$\forall$element $|\{S : e \in S| = d$, then $\text{OPT}_f = \dfrac{n}{d}$.

Hint: weak duality.

3) Find an example in which

$$\mathrm{OPT} \geq \Omega(\lg n) \cdot \mathrm{OPT}_f \, .$$

Hint : Elements $\mathcal{U} = \{0,1\}^K - 0^K$ (all nonzero strings)

Sets $\{S_i\}_{i \in \mathcal{U}}$

where

$$S_i = \{ u \in \mathcal{U} \mid \sum_{j=1}^{K} u_j \cdot i_j \text{ is odd} \}$$

show : $\forall i : \quad |S_i| = 2^K/2$

$\forall u : \quad |\{S_i : u \in S_i\}| = 2^K/2$

use prob. (2) to prove $\mathrm{OPT}_f \leq 2$.

prove $\mathrm{OPT} \geq K$ (system of $<K$ equations in GF(2) has nonzero solution)

Problems

4) Suppose that $\forall$ element : $|\{S : e \in S\}| \leq d$.

Find an approximation algorithm with

$$|\mathrm{ALG}| \leq d \cdot \mathrm{OPT} \, .$$

5) Improve the analysis of the approximation ratio of the Greedy algorithm if

$\forall$ Set : $|\mathrm{Set}| \leq K$. (say, $K = 3$).

(what if $K = 2$???)

Problems

6) Weak duality

Primal	Dual
min $c^t x$	max $y^t b$
s.t. $Ax \geq b$	s.t $y^t A \leq c$
$x \geq 0$	$y \geq 0$

prove that: if x is a primal solution & y is a dual solution, then $c^t x \geq y^t b$.

Encore : Rounding a fractional set-cover

Suppose $\{x_S\}$ is an optimal fractional set-cover,

I.E., $\forall e : \sum_{e \in S} x_S \geq 1$ and $\sum_S x_S = OPT_f$.

How can we obtain an integral set-cover $\hat{x}_S \in \{0,1\}$ from x_S ?

2 Goals :

feasibility : $\forall e \ \exists S : e \in S \ \& \ \hat{x}_S = 1$

cost : $\dfrac{\sum_S \hat{x}_S}{\sum_S x_S}$ is small $\left(\sum_S \hat{x}_S \leq O(\lg n) \cdot OPT_f \right)$

Randomized Rounding

For each set S, randomly flip a "coin" $\hat{x}_S$ such that $\Pr[\hat{x}_S = 1] = x_S$.

Now: $\forall$ element e:

$$\Pr[e \text{ is covered}] = \Pr[\exists S: e \in S \ \&\ \hat{x}_S = 1]$$

$$= 1 - \Pr[\forall S: e \in S \Rightarrow \hat{x}_S = 0]$$

$$= 1 - \prod_{S: e \in S} (1 - x_S)$$

$e^{-x} \geq 1 - x$

$$\geq 1 - e^{-\sum_{e \in S} x_S} \geq 1 - e^{-1} \approx 0.632$$

$\Rightarrow$ Each element is covered with constant prob.

Repeat this experiment $\ln(4n)$ times

(S is in the cover if $\hat{x}_S = 1$ in one of experiments)

$$\Pr(e \text{ is not covered}) \leq \left(\frac{1}{e}\right)^{\ln(4n)} = \frac{1}{4n}$$

$$\Rightarrow \Pr(\exists e \text{ not covered}) \leq \sum_e \Pr(e \text{ not covered}) = \frac{1}{4}$$

$\Rightarrow$ with prob $\leq \frac{1}{4}$ every element is covered if we repeat $\ln(4n)$ times.

Amplify prob. of success by repeating $k \cdot \ln(4n)$ times $\Longrightarrow$ $\Pr(\text{Success}) \geq 1 - \left(\frac{1}{4}\right)^k$.

What about cost?

$$E\left[\sum_S \hat{x}_S\right] = \sum_S E[\hat{x}_S] = \sum_S x_S$$

We repeat $\ln(4n)$ times $\Longrightarrow E(\text{cover}) = \ln(4n) \cdot OPT_f$.

Markov Inequality:

$$\Pr(\text{cover} \geq 4 \cdot \ln(4n)\, OPT_f) \leq \frac{1}{4}.$$