Hitting Sets Online

Guy Even - Tel-Aviv Univ.

joint work with

Shakhar Smorodinsky

- Ben-Gurion Univ.

Outline

1) Define: online hitting set problem

2) Characterization of competitive ratio by unique-max colorings (for special hypergraphs)

3) Applications

4) Open problems...

Definitions

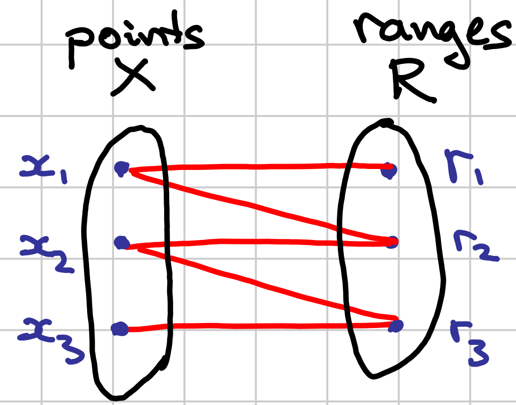
* Hypergraph (X, R)

* $S \subseteq X$ stabs $r \in R$ if

$$S \cap r \neq \emptyset$$

* $S \subseteq X$ is a hitting set if

  S stabs every range



$r_1 = \{x_1\}$

$r_2 = \{x_1, x_2\}$

$r_3 = \{x_2, x_3\}$

* Minimum Hitting Set Problem (Min-HS):

  find a hitting set with the smallest cardinality.

Hitting Sets Online

* input : set of points X

* adversary: introduces a seq. $\sigma = \{r_i\}_{i=1}^{s}$ of ranges (one-by-one).

  Let $\sigma_i \triangleq \{r_1, r_2, \dots, r_i\}$

* algorithm: upon arrival of r_i, adds a point to $Alg(\sigma_{i-1})$, if needed, so that r_i is stabbed by $Alg(\sigma_i)$

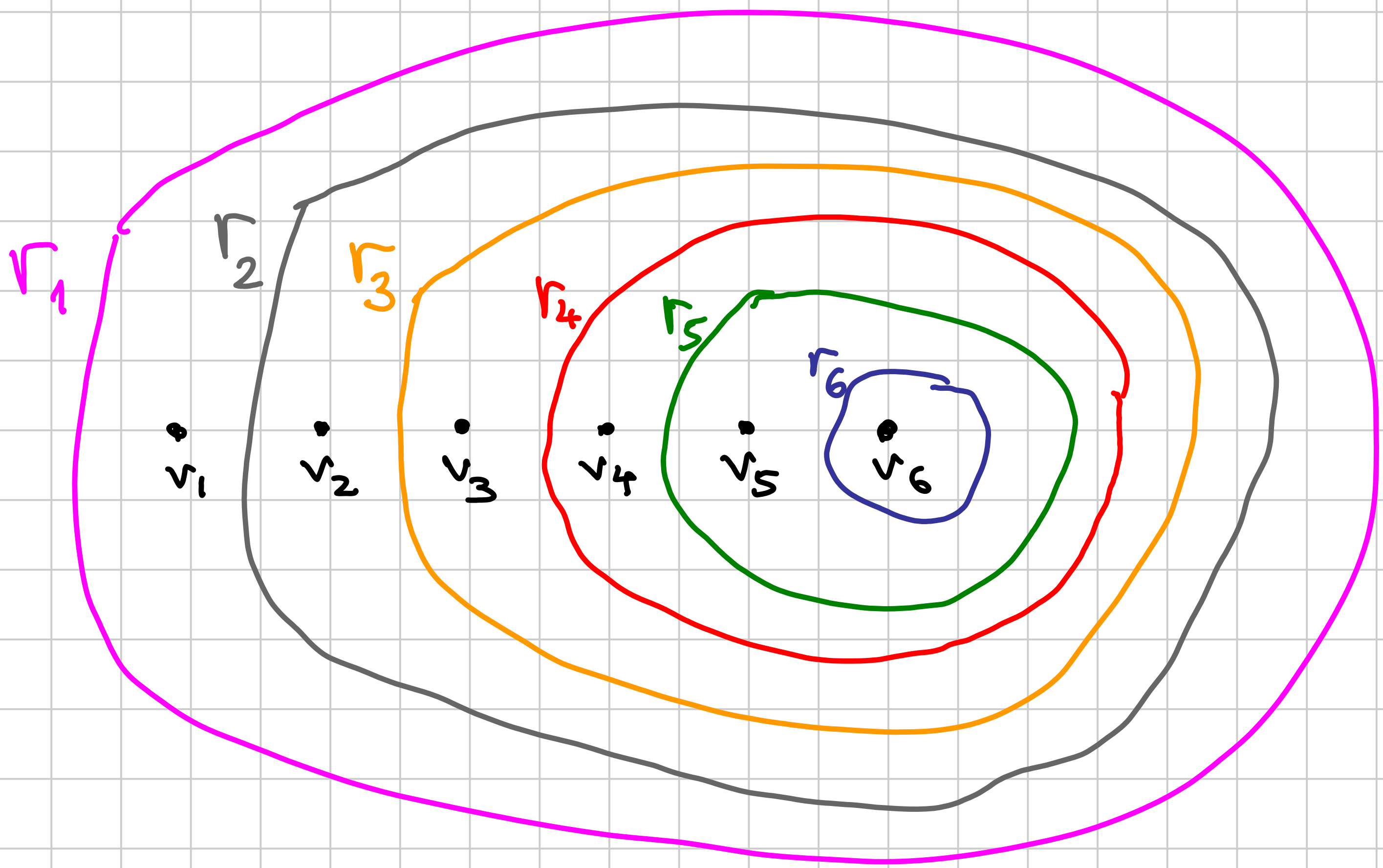
Example:

Sequence of ranges: $\{r_1, r_2, \dots, r_6\}$

Alg. stabs r_i by v_i

$|Alg| = 6$

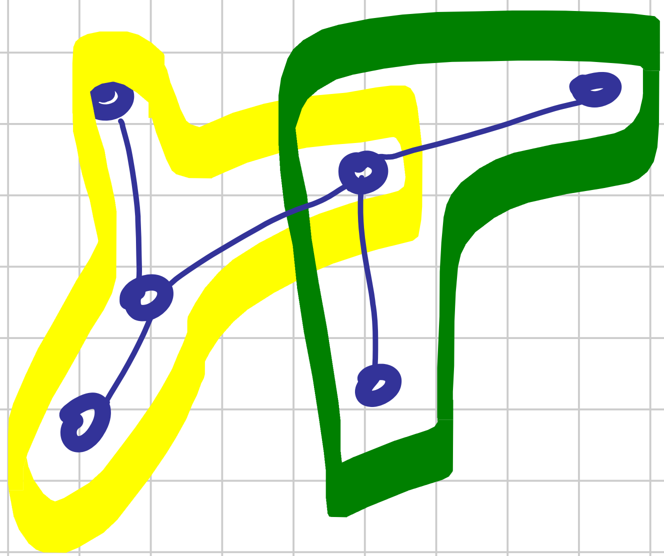
$|OPT| = 1$

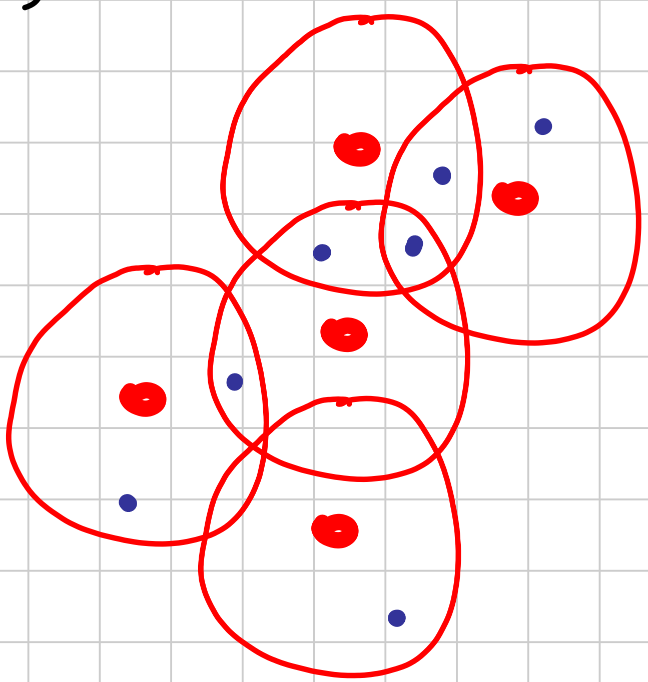


Applications

1) servers in a VPN (connected subgraphs)

2) base stations in a wireless network (unit disks)





Competitive Ratio

The competitive ratio of a deterministic HS algorithm is:

$$\rho_{ALG}(H) \triangleq \sup_{\sigma} \frac{|Alg(\sigma)|}{|Min\text{-}HS(\sigma)|}$$

$$\rho(H) \triangleq \inf \left\{ \rho_{ALG}(H) \mid ALG \right\}$$

Can we characterize $\rho(H)$ & design online algs?

Previous Results

Offline HS – General Case

NP-C (Karp 72)

(R = Ranges)

hard to approx better than $\ln|R|$ (Feige 98, RS)

Greedy alg $H_{|R|} \leq 1 + \ln|R|$ approx (J74, C79, L)

Offline HS – Geometric Settings

PTAS for unit disks (HM85, G)

ε-nets : $O(\frac{1}{\varepsilon})$ ε-net $\Rightarrow$ $O(1)$-apx [BG, ERS]

[CV-07] extend $o(\log n)$ for fat-Δ, $O(1)$ cubes....

[AES-09] rectang. ... [MR-09] Local search ! PTAS $\frac{1}{2}$-spaces

Previous Results -(cont)

Online Set Cover (Alon, Azar, Buchbinder, Naor)

$O(\log |X| \cdot \log |R|)$ - comp. ratio

* [AABN] considered equiv. Set Cover (elements arrive online need to covered by sets.

* Primal-Dual alg computes frac solution (online) followed by rounding.

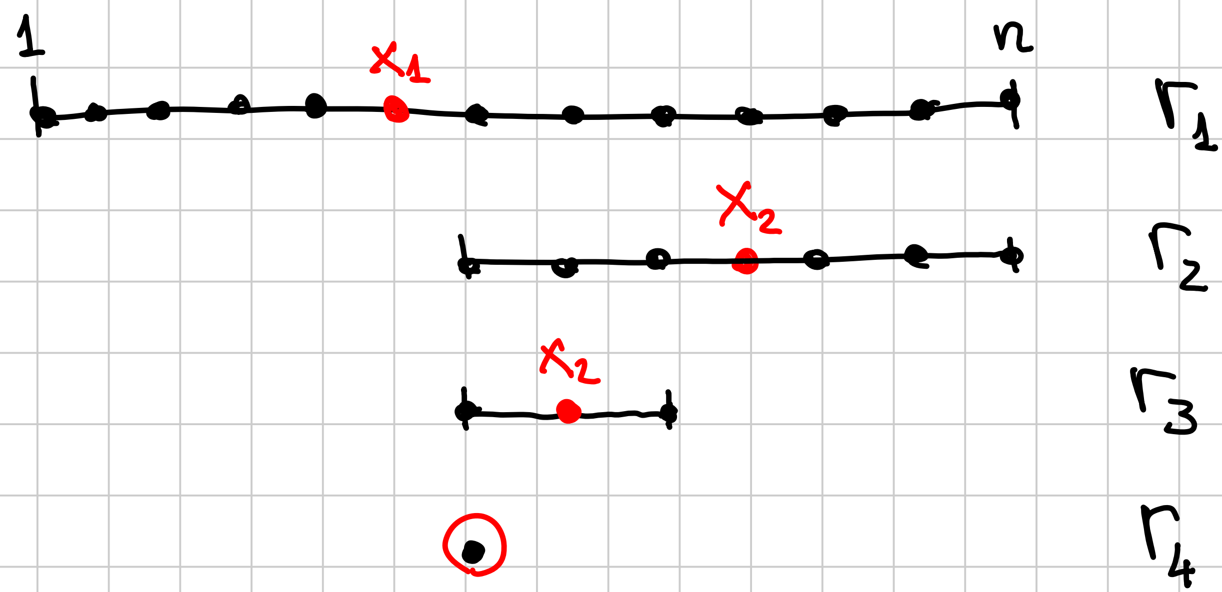
Example: Intervals

$X \triangleq \{1, 2, \dots, n\}$

$R \triangleq \{ [i,j] \mid i < j \}$

Claim: $\rho(X, R) = \lfloor \lg n \rfloor + 1$

proof: $(\geq)$



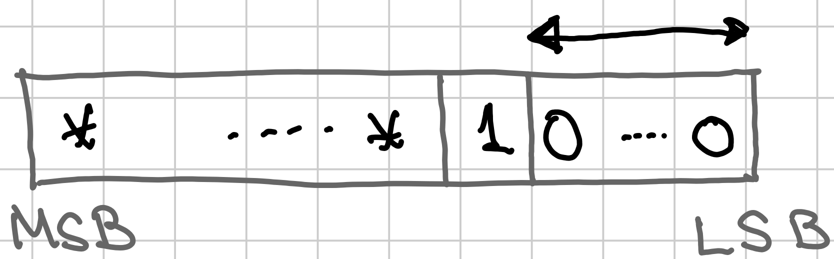
Example: Intervals (cont)

$X \triangleq \{1, 2, \ldots, n\}$

$R \triangleq \{ [i, j] \mid i < j \}$

Claim: $\rho(X, R) = \lfloor \lg n \rfloor + 1$

proof: $(\leq)$ If r_i is not stabbed, alg picks $x_i \in r_i$ with longest run of zeros:



Example: Intervals (cont)

$X \triangleq \{1, 2, \ldots, n\}$

$R \triangleq \{ [i,j] \mid i < j \}$

Claim: $\rho(X, R) = \lfloor \lg n \rfloor + 1$

Compare with [AABN]

Comp. ratio $= O(\log^2 n)$

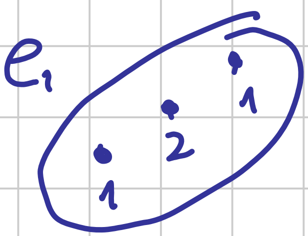
Unique Max Coloring (UM)

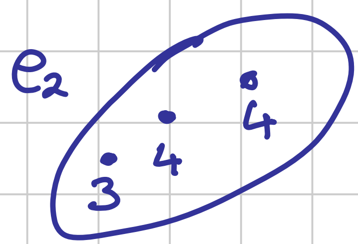
for a coloring $c : X \to \mathbb{N}$, define

$$\forall r \in R: \quad c_{max}(r) \triangleq \max\{c(x) \mid x \in r\}$$

DEF: $c : X \to \mathbb{N}$ is a UM coloring if

$$\forall r \in R: \quad |\{x \in R : c(x) = c_{max}(r)\}| = 1$$





$c_{max}(e_1) = 2$, unique

$c_{max}(e_2) = 4$, not unique!

Unique Max Coloring

(max color in each range appear only once)

* If H is a graph then

UM-coloring $\equiv$ proper-coloring.

DEF: $\chi_{UM}(H) \triangleq \min\{\#\text{colors in } C : C \text{ is a UM-coloring of } H\}$

I-Type hypergraphs

DEF: a hypergraph $H=(V,E)$ is I-type if

$$\forall e_1, e_2 \in E: \quad e_1 \cap e_2 \neq \emptyset \Rightarrow e_1 \cup e_2 \in E$$

Example: intervals on a line

Main Result

Thm: If $H=(V,E)$ is I-type, then

$$\rho(H) = \chi_{um}(H).$$

proof: Reductions.

$\rho(H) \leq \chi_{um}(H)$: given a um-coloring, describe an online alg.

$\rho(H) \geq \chi_{um}(H)$: given an online alg, describe a um-coloring.

$$\rho(H) \le \chi_{um}(H)$$

* ALG stabs a new unstabbed range r_i by the vertex $v \in r_i$ with highest color.

* $ALG(\sigma, j) \triangleq \left\{ r_i \in \sigma \;\middle|\; \begin{array}{l} r_i \text{ first stabbed by} \\ \text{a vertex colored } j \end{array} \right\}$

* claim: $\forall e_1, e_2 \in ALG(\sigma, j)$: $e_1 \cap e_2 = \emptyset$

* $|ALG(\sigma)| = \sum_j |ALG(\sigma, j)|$

$$\le \chi_{um}(H) \cdot |OPT(\sigma)|$$

$$\rho(H) \geq \chi_{um}(H)$$

properties of I-type hypergraphs:

1) maximal ranges (wrt inclusion) are disjoint.

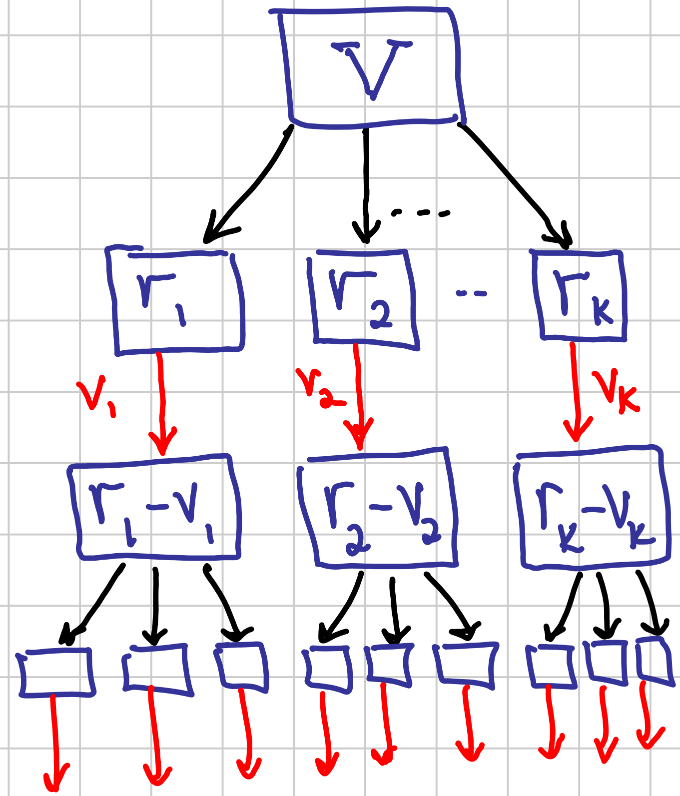
2) $\forall x \in X$ $\exists$ unique maximal range $r \in R$: $x \in r$.

3) Projection of I-type hypergraph is also I-type.

$$(V \setminus U, \{e \setminus U\}_{e \in E})$$

$$\rho(H) \geq x_{um}(H)$$

Partition V into maximal ranges



$v_i \leftarrow \text{ALG}(r_i)$

Partion each $r_i - v_i$.

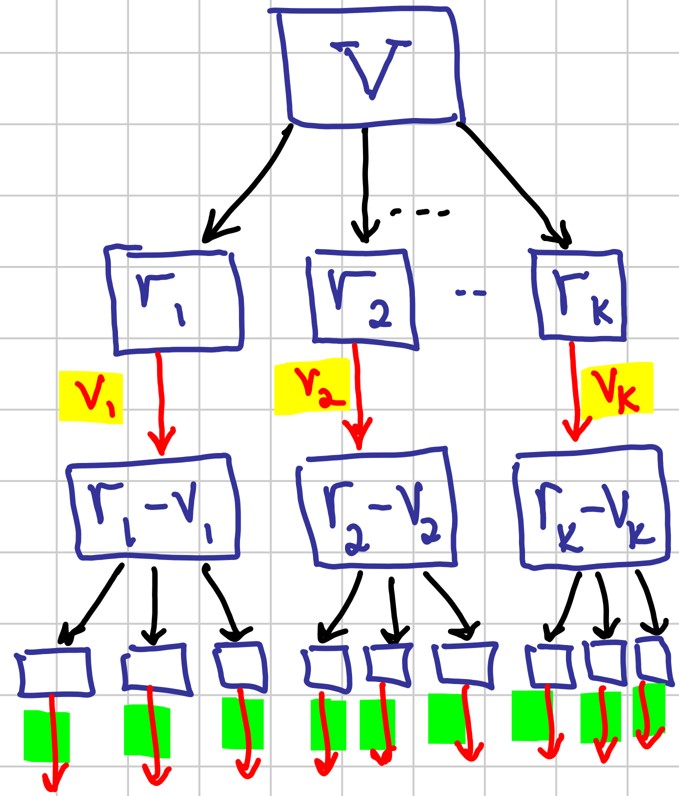
for each maximal range

$r_{i,j}$ of $r_i - v_i$: $\quad v_{ij} \leftarrow \text{ALG}(\underbrace{\{r_i\}}_{\text{history}}, \underbrace{r_{ij}}_{\text{query}})$

⋮

$$\rho(H) \geq \chi_{um}(H)$$

* Color first level by $\rho(H)$, next by $\rho(H)-1$, etc.



* UM-coloring because max ranges are disjoint.

* at most $\rho(H)$ colors.

Otherwise, path $> \rho(H)$ nested ranges with $ALG(\text{path}) > \rho(H)$.

Q.E.D.

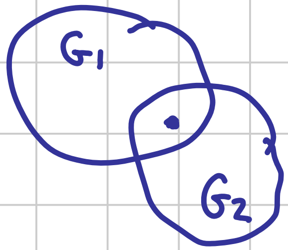
Application: connected subgraphs

a graph $G = (V, E)$

$X = V$

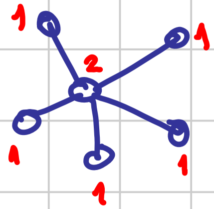
R = connected subgraphs of G

$H(X, R)$ is I-type



So, $\rho(H) = \chi_{um}(H)$!

Implications

1) Star 

: $\rho(\text{star}) = \chi_{um}(\text{star}) = 2$

$|R| = 2^{n-1} \Rightarrow$ [AABN] give $O(n)$ (note: VC-dim $= n-1$)

2) simple path : $\rho(\text{path}) = \Theta(\log n)$

[AABN] gives $O(\lg^2 n)$ because $|R| = \Theta(|X|^2)$

3) Tree : $\rho(\text{tree}) = \Theta(\log(\text{diameter}))$

[AABN] gives $O(n)$

Implications - cont

4) planar graph : $\rho(\text{planar}) = O(\sqrt{n})$

again [AABN] is $O(n)$

5) tree width d : $\rho(tw) = O(d \log n)$

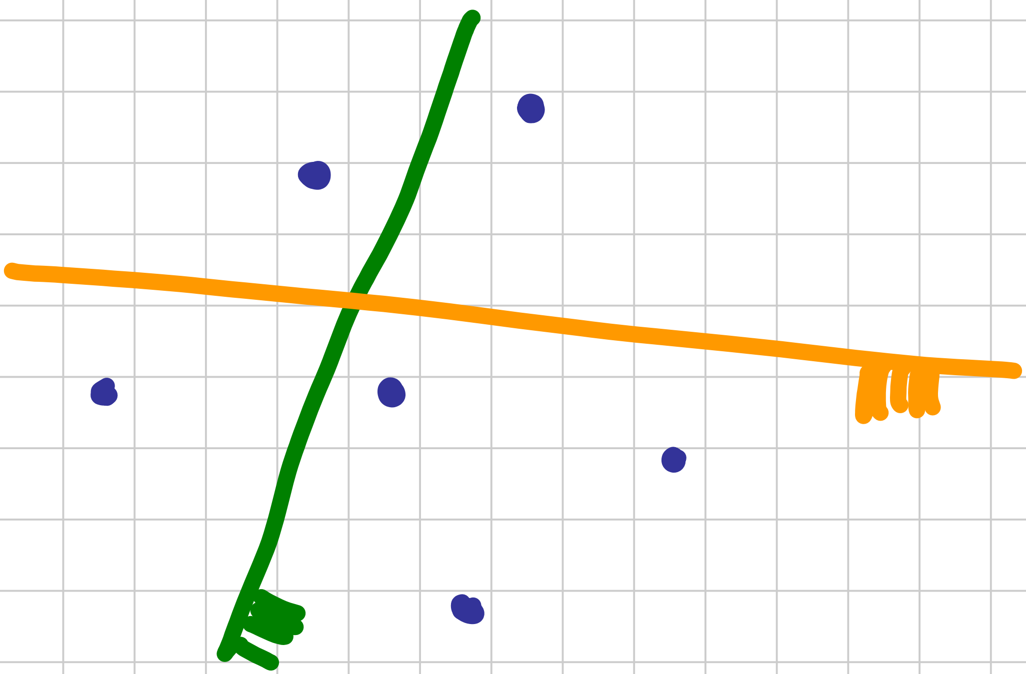
again [AABN] is $O(n)$

What About Geometry?

points & half-planes:

$X \subseteq \mathbb{R}^2$

$R = \{ r \subseteq X : r$ is an intersection of X with a half-plane $\}$



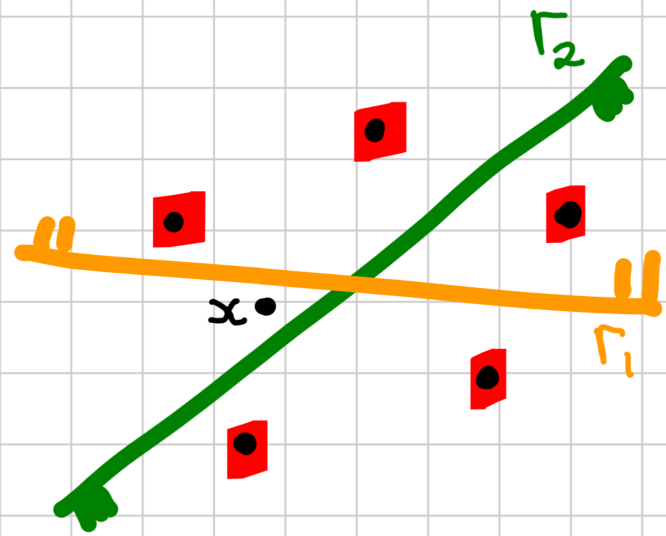
Thm: $\exists$ online alg with comp. rat. $O(\lg n)$

$\forall$ online alg: comp. rat. $\Omega(\lg n)$

Points & Half Planes

$H(X, R)$ is NOT I-type

$r_1 \cup r_2 = X - \{x\}$

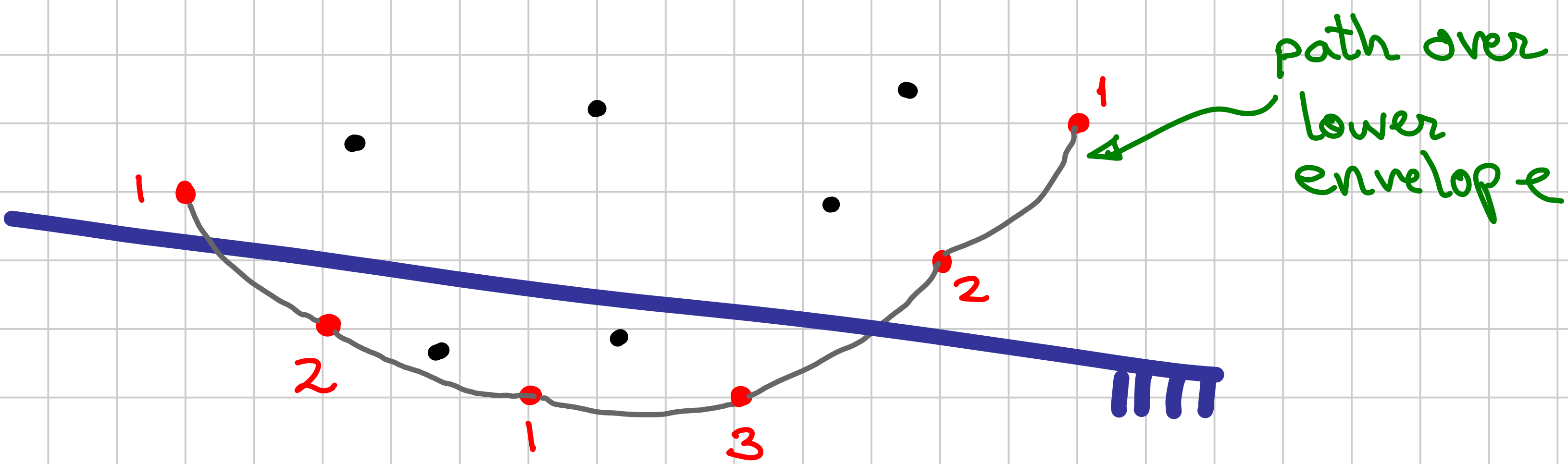


Online Alg: points & half planes

1) suffice to consider extreme pts in X.
(separate: lower & upper envelopes)

2) $\forall r$: $\quad r \cap \text{extreme}(X) = \text{interval}$

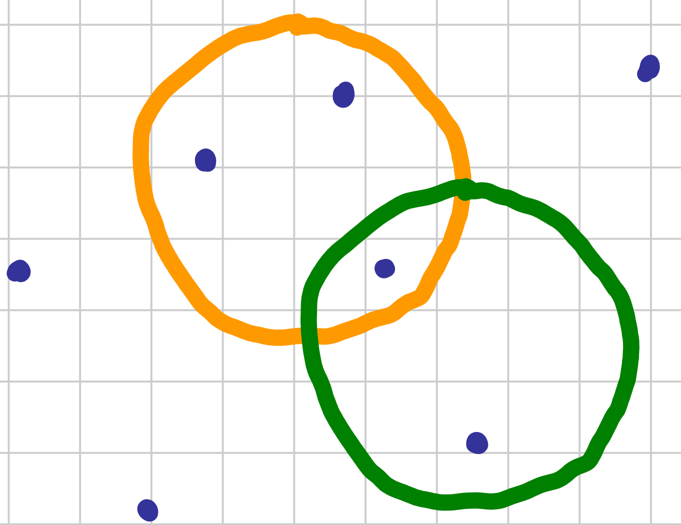
$\Longrightarrow$ reduction to intervals



Application: points & unit disks

$X \subseteq \mathbb{R}^2$

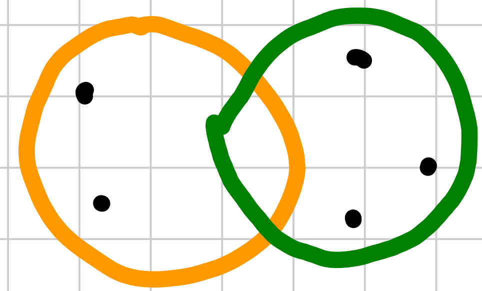
$R = \{ r \subseteq X : r \text{ is an intersection of } X \text{ with a unit disk} \}$



Thm: $\exists$ online alg with comp. rat. $O(\lg n)$

$\forall$ online alg: comp. rat. $\Omega(\lg n)$

points & unit disk



* again not I-type
* alg uses: tiling

  defines extreme pts in each tile
* UM-color each "path" of extreme pts
* stabs disk by highest color extreme pt.

OPEN PROBLEM

n points & disks (not unit disks).

$$|R| = \Theta(|X|^3)$$

$\Rightarrow$ [AAABN] give $O(\lg^2 n)$-comp. ratio.

Question: design an online alg with $O(\log n)$ comp. ratio.

More open questions

1) weights (min weight HS)

2) rand. online alg. ($\chi_{um} \overset{?}{\leq} \rho_{RAND}(H)$)

3) what if hypergraph is not I-type ?

clearly $\rho(H) \leq \chi_{um}$(completion of H)

but not tight ...

Hypergraphs that are not I-type

1) Vertex Cover over a clique ($R = E(K_n)$)

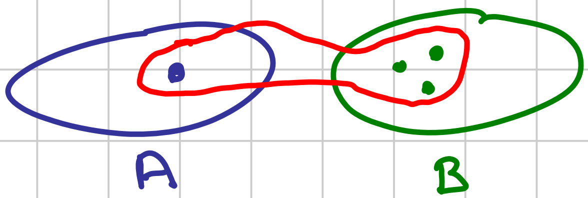
$\rho(H) = 2$ (pick both ends of unstabbed edge)

$\chi_{um}(H) = \chi(K_n) = n$

2) $X \stackrel{\Delta}{=} A \dot{\cup} B$

$R \stackrel{\Delta}{=} \{ r \subseteq X \mid |r| = 3, |r \cap A| = 1, |r \cap B| = k-1 \}$

$\chi_{um}(X, R) = 2$



But: $\rho(H) = k$

What is Vertex Ranking? [KMS-95]

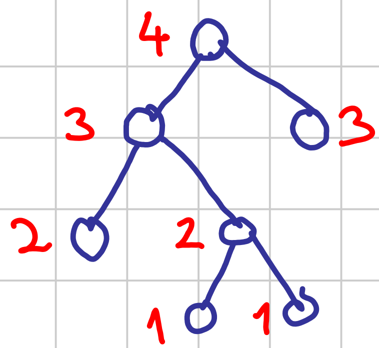
$c : V \to \mathbb{N}^+$ is a vertex ranking if

$\forall u, v \in V : \quad c(u) = c(v)$

$\Longrightarrow$

$\forall u \overset{\text{path}}{\rightsquigarrow} v \quad \exists w \in \text{path} : \quad c(w) > c(u)$.

example:



$c(u) \triangleq \text{height}(u)$

$$c_{max}(V) \triangleq \max \{ c(v) \mid v \in V \}$$

Questions:

1) $\exists (X, R)$: comp. ratio $= |X|$?

2) $\exists (X, R)$: $\log |R| = \Theta(|X|)$
with comp. ratio $= o(|X|)$?

3) constant comp. ratio $\Longrightarrow$ const. VC-dim ?