Covering Graphs Using Trees and Stars

Guy Even (Tel-Aviv), Naveen Garg (Delhi),

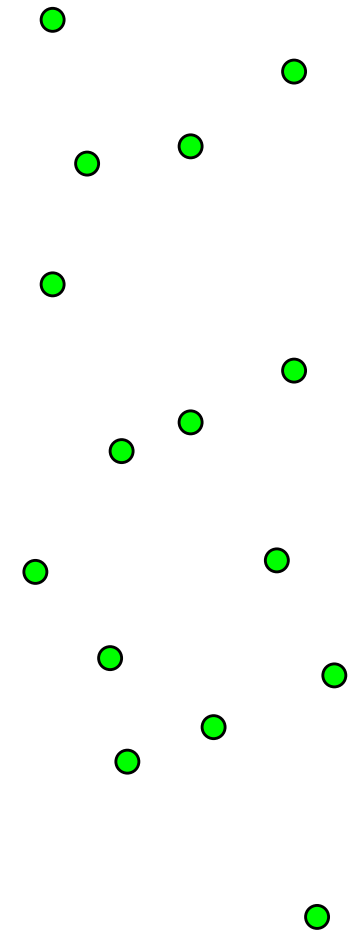
and

Jochen Könemann, R. Ravi and A. Sinha (Pittsburgh)

Third Haifa Workshop on Interdisciplinary Applications of Graph Theory, Combinatorics and Computing

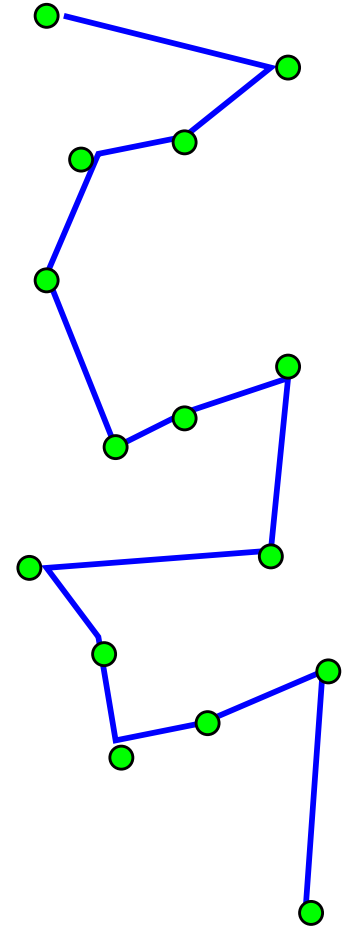
Motivation

- Consider a TSP instance with a large optimal tour, e.g. $w(tour^*) > 1000$.



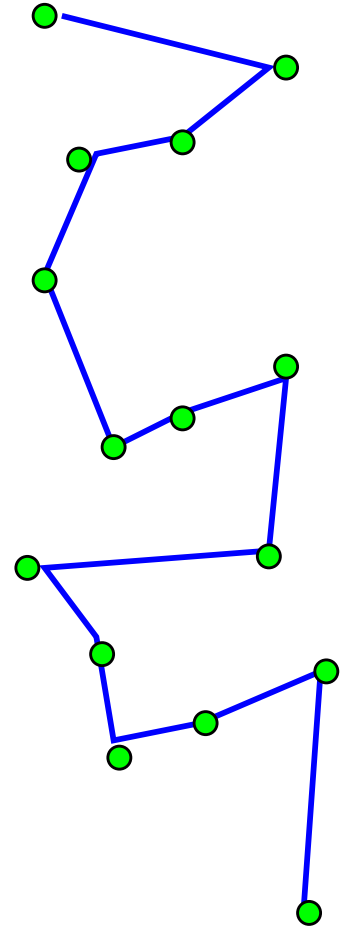
Motivation

- Consider a TSP instance with a large optimal tour, e.g. $w(tour^*) > 1000$.
- Suppose regulation dictates: agent may travel at most 100 km.



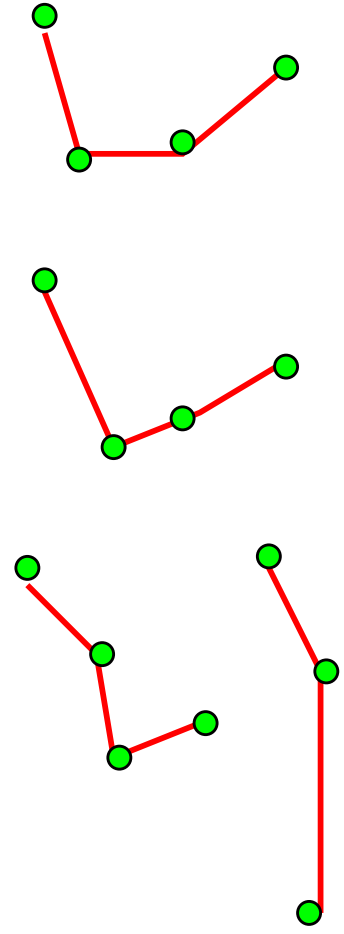
Motivation

- Consider a TSP instance with a large optimal tour, e.g. $w(tour^*) > 1000$.
- Suppose regulation dictates: agent may travel at most 100 km.
- $\Rightarrow$ must employ multiple agents.



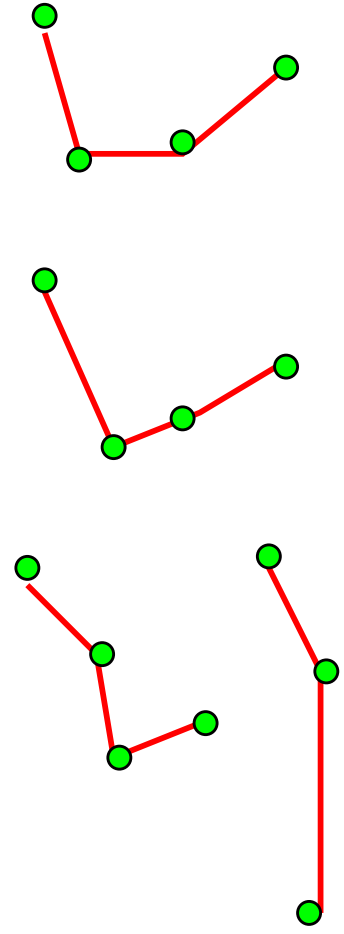
Motivation

- Consider a TSP instance with a large optimal tour, e.g. $w(tour^*) > 1000$.
- Suppose regulation dictates: agent may travel at most 100 km.
- $\Rightarrow$ must employ multiple agents.
- $\Rightarrow$ k-traveling salespeople problem.



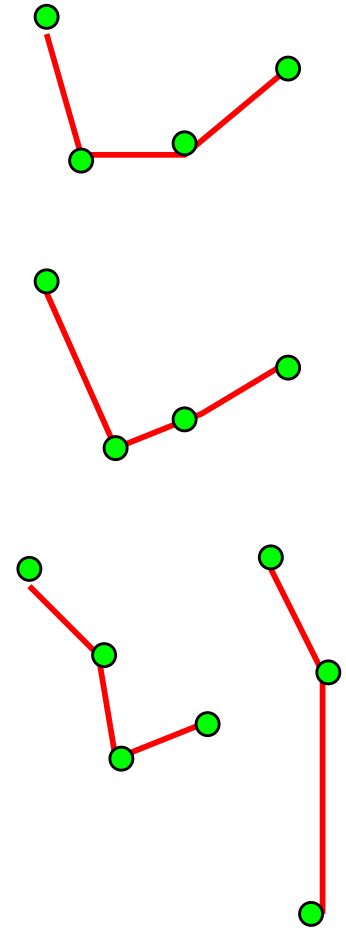
Motivation

- Consider a TSP instance with a large optimal tour, e.g. $w(tour^*) > 1000$.
- Suppose regulation dictates: agent may travel at most 100 km.
- $\Rightarrow$ must employ multiple agents.
- $\Rightarrow$ k-traveling salespeople problem.
  - Cover the vertices of a graph with k tours.



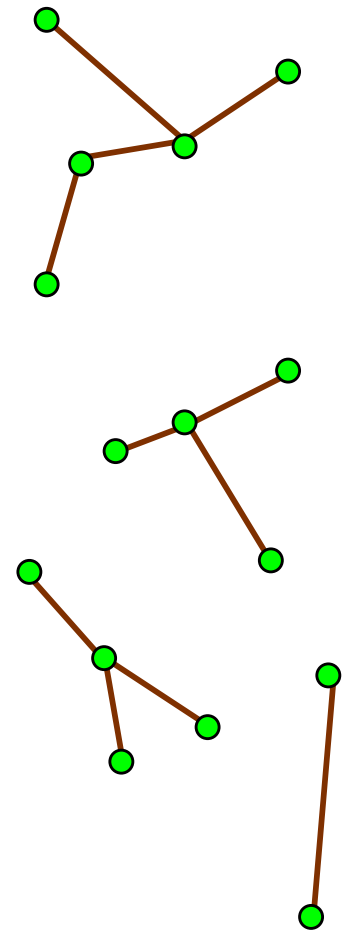
Motivation

- Consider a TSP instance with a large optimal tour, e.g. $w(tour^*) > 1000$.
- Suppose regulation dictates: agent may travel at most 100 km.
- $\Rightarrow$ must employ multiple agents.
- $\Rightarrow$ k-traveling salespeople problem.
  - Cover the vertices of a graph with k tours.
  - Balance the load of the agents: minimize the maximum tour.



Motivation

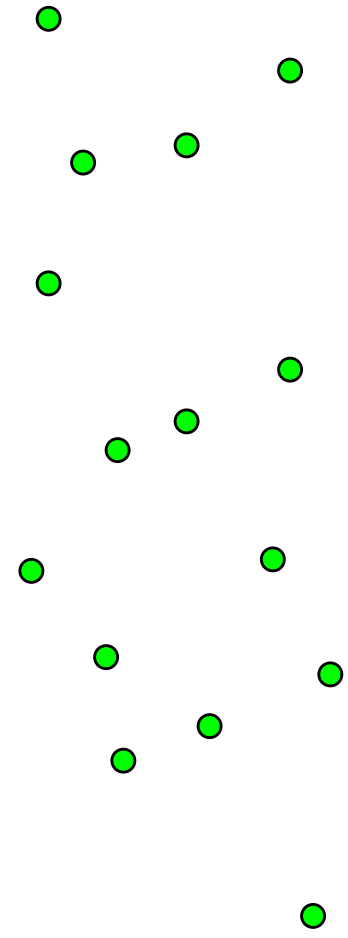
- Consider a TSP instance with a large optimal tour, e.g. $w(tour^*) > 1000$.
- Suppose regulation dictates: agent may travel at most 100 km.
- $\Rightarrow$ must employ multiple agents.
- $\Rightarrow$ k-traveling salespeople problem.
  - Cover the vertices of a graph with k tours.
  - Balance the load of the agents: minimize the maximum tour.
- MST is a constant ratio approx of a min tour $\Rightarrow$ k-Tree Cover Problem.



k-Tree Cover Problem

k-Tree Cover Problem

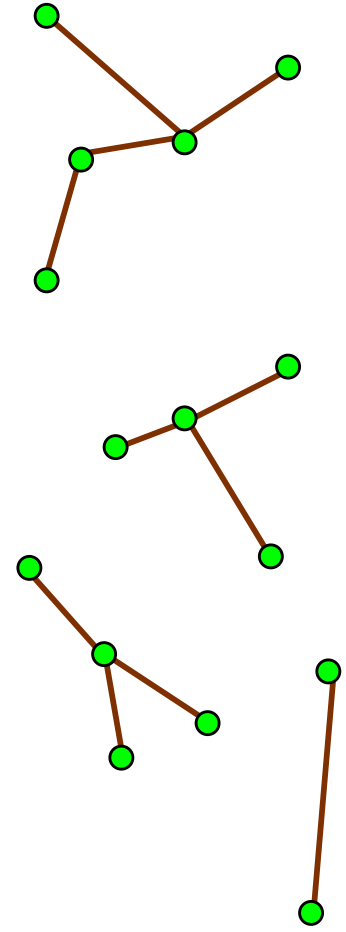
Input: (i) integer k and (ii) $G = (V, E)$ - an undirected graph with positive integral edge weights $w : E \rightarrow \mathbb{N}^+$.



k-Tree Cover Problem

Input: (i) integer k and (ii) $G = (V, E)$ - an undirected graph with positive integral edge weights $w : E \rightarrow \mathbb{N}^+$.

k-tree cover: a set $\mathcal{T}$ of trees $\{T_i\}_i$ such that $V = \bigcup_{i=1}^{k} V(T_i)$.

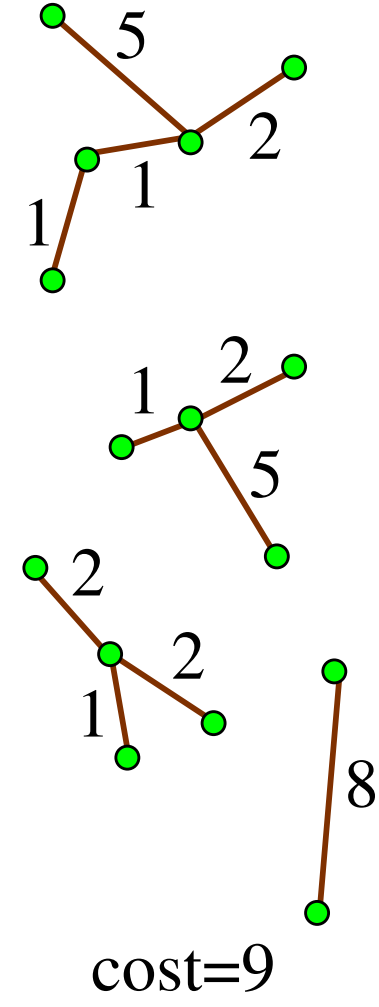


k-Tree Cover Problem

Input: (i) integer k and (ii) $G = (V, E)$ - an undirected graph with positive integral edge weights $w : E \rightarrow \mathbb{N}^+$.

k-tree cover: a set $\mathcal{T}$ of trees $\{T_i\}_i$ such that $V = \bigcup_{i=1}^{k} V(T_i)$.

cost : $\text{cost}(\mathcal{T}) = \max_{T_i \in \mathcal{T}} w(T_i)$.



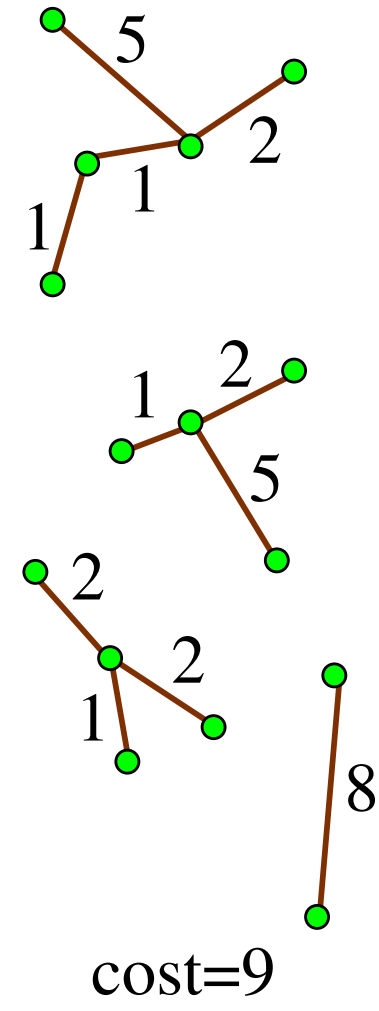
k-Tree Cover Problem

Input: (i) integer k and (ii) $G = (V, E)$ - an undirected graph with positive integral edge weights $w : E \rightarrow \mathbb{N}^+$.

k-tree cover: a set $\mathcal{T}$ of trees $\{T_i\}_i$ such that $V = \bigcup_{i=1}^{k} V(T_i)$.

cost : $\text{cost}(\mathcal{T}) = \max_{T_i \in \mathcal{T}} w(T_i)$.

goal : find a minimum cost k-tree cover.



k-Tree Cover Problem

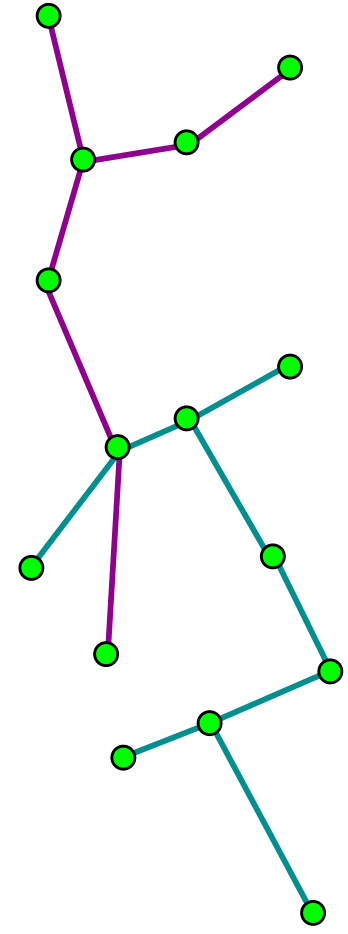
Input: (i) integer k and (ii) $G = (V, E)$ - an undirected graph with positive integral edge weights $w : E \rightarrow \mathbb{N}^+$.

k-tree cover: a set $\mathcal{T}$ of trees $\{T_i\}_i$ such that $V = \bigcup_{i=1}^{k} V(T_i)$.

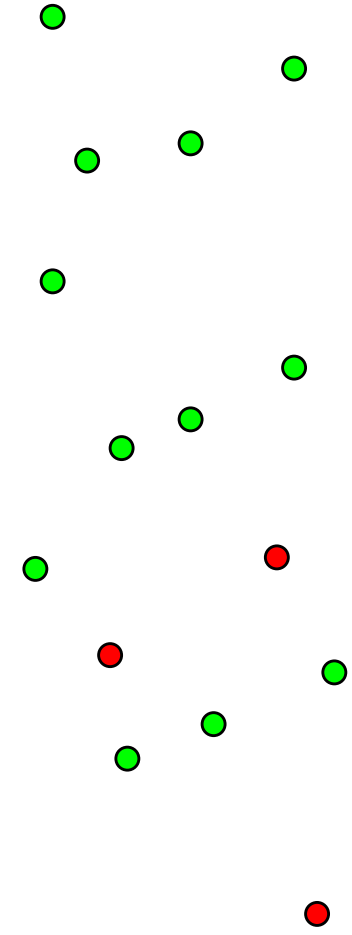
cost : $\text{cost}(\mathcal{T}) = \max_{T_i \in \mathcal{T}} w(T_i)$.

goal : find a minimum cost k-tree cover.

remark : trees may share nodes & edges in a tree cover.



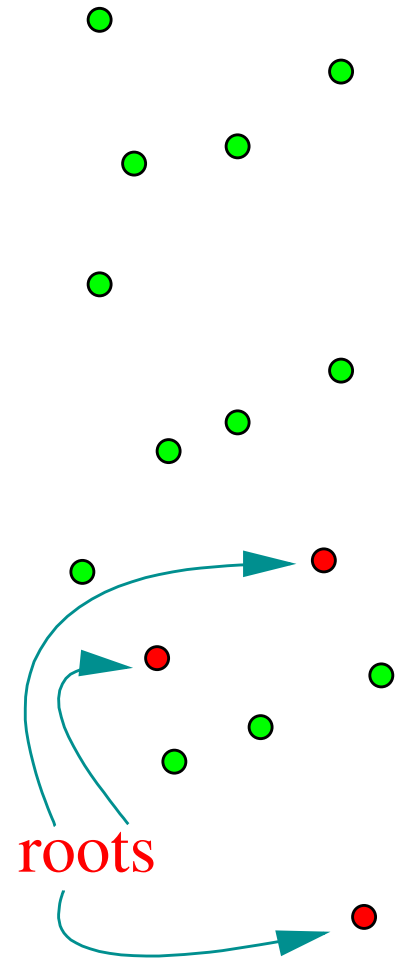
k-Tree Cover Problem: the rooted case



k-Tree Cover Problem: the rooted case

Roots: Input contains also a set of roots:

$$R = \{r_1, r_2, \ldots, r_k\}.$$



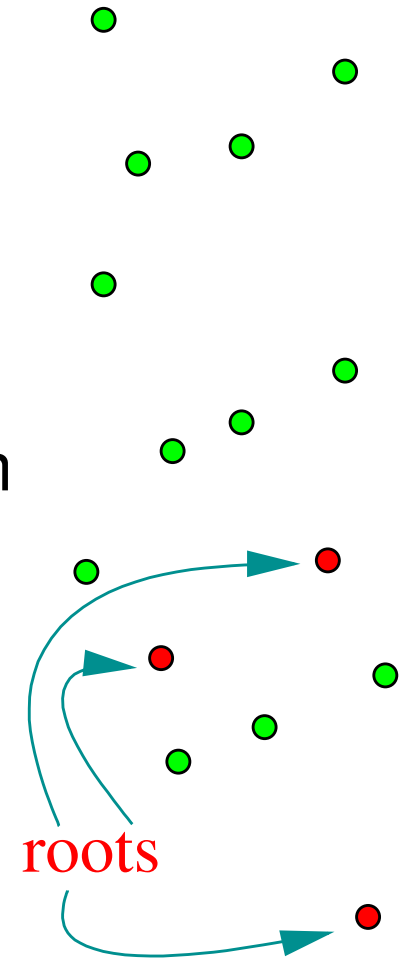
k-Tree Cover Problem: the rooted case

Roots: Input contains also a set of roots:

$$R = \{r_1, r_2, \ldots, r_k\}.$$

k-rooted tree cover: a k-tree cover $\{T_i\}_i$ such that

$$\forall i \ : \ r_i \in T_i.$$



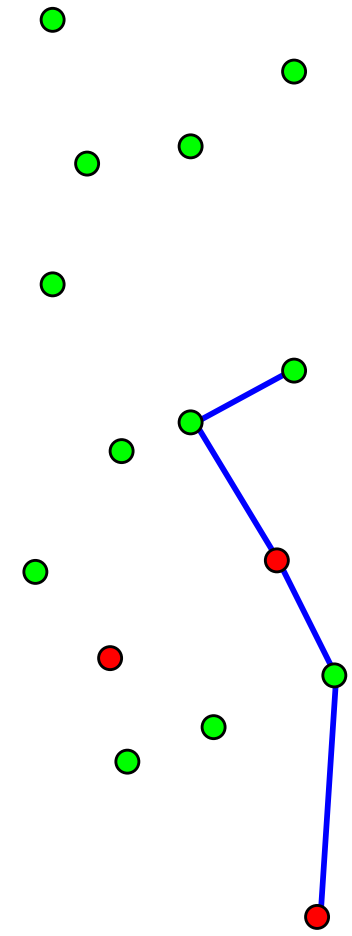
k-Tree Cover Problem: the rooted case

Roots: Input contains also a set of roots:

$$R = \{r_1, r_2, \ldots, r_k\}.$$

k-rooted tree cover: a k-tree cover $\{T_i\}_i$ such that

$$\forall i \ : \ r_i \in T_i.$$



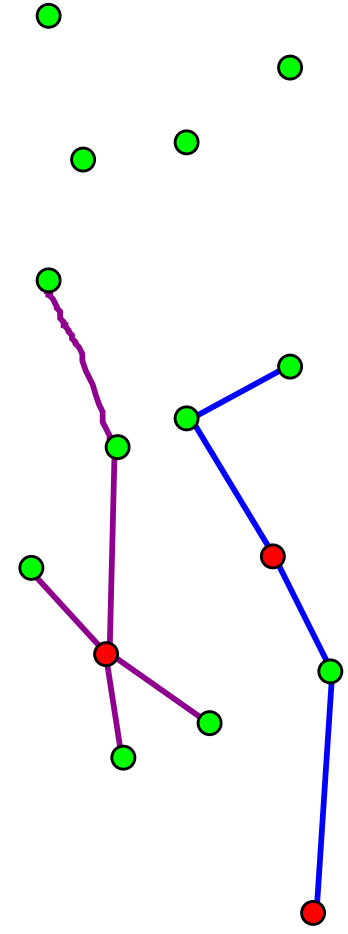
k-Tree Cover Problem: the rooted case

Roots: Input contains also a set of roots:

$$R = \{r_1, r_2, \ldots, r_k\}.$$

k-rooted tree cover: a k-tree cover $\{T_i\}_i$ such that

$$\forall i \ : \ r_i \in T_i.$$



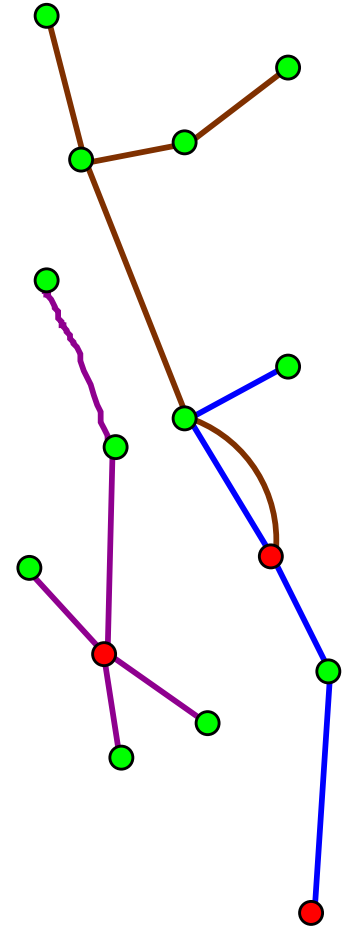
k-Tree Cover Problem: the rooted case

Roots: Input contains also a set of roots:

$$R = \{r_1, r_2, \ldots, r_k\}.$$

k-rooted tree cover: a k-tree cover $\{T_i\}_i$ such that

$$\forall i \ : \ r_i \in T_i.$$



k-Tree Cover Problem: the rooted case

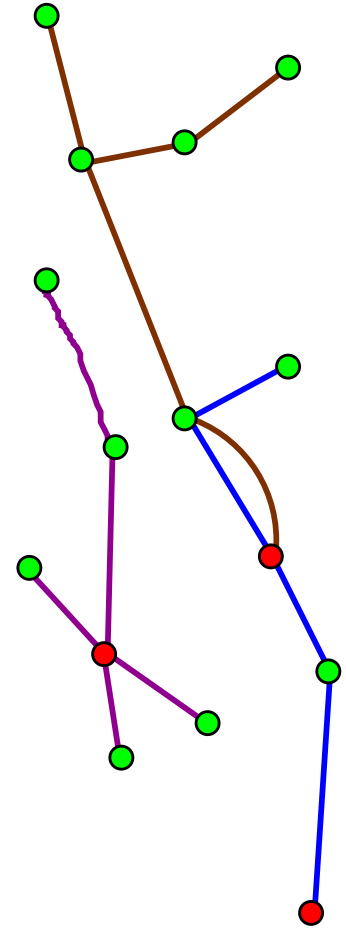
Roots: Input contains also a set of roots:

$$R = \{r_1, r_2, \ldots, r_k\}.$$

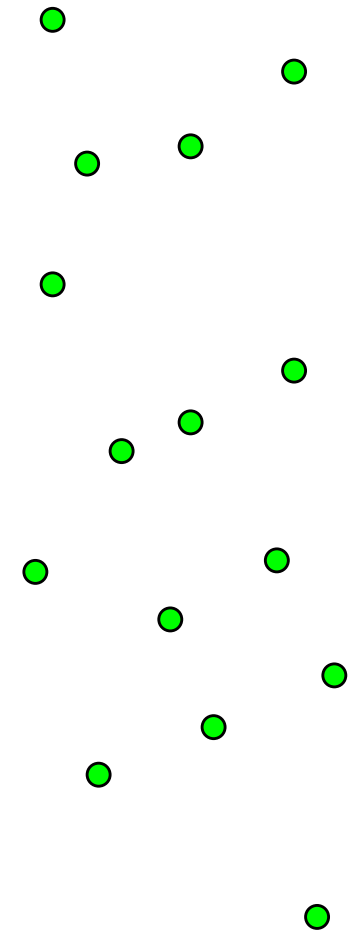
k-rooted tree cover: a k-tree cover $\{T_i\}_i$ such that

$$\forall i \ : \ r_i \in T_i.$$

motivation : agents start their tour in different locations.



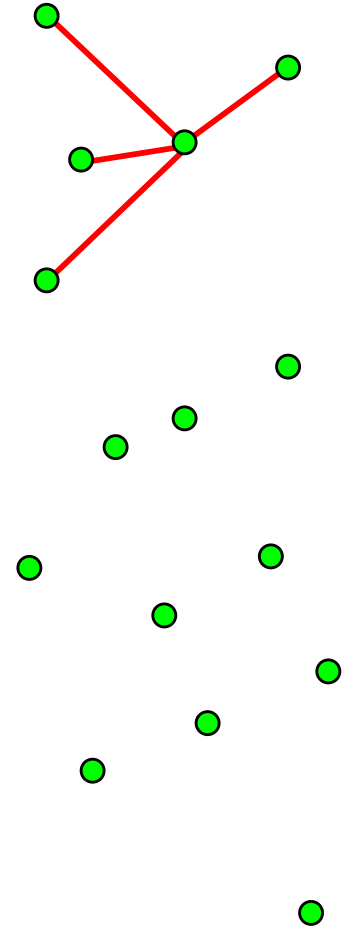
Star Covers



Star Covers

k-Star Cover: a k-tree cover $\{T_i\}_i$ in which

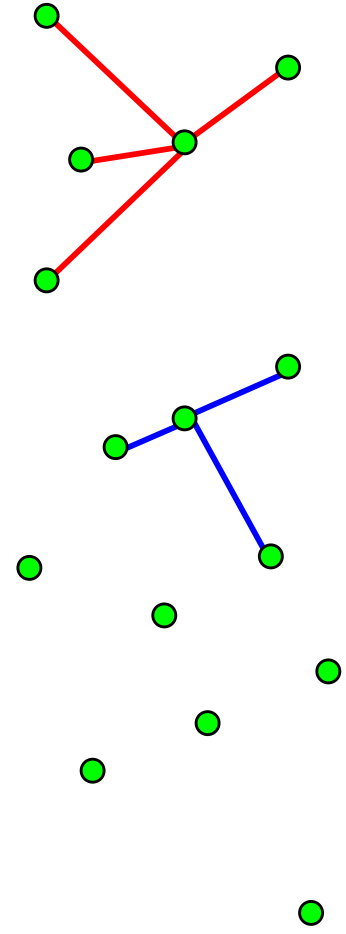
$$\forall i \ : \ T_i \text{ is a star}$$



Star Covers

k-Star Cover: a k-tree cover $\{T_i\}_i$ in which

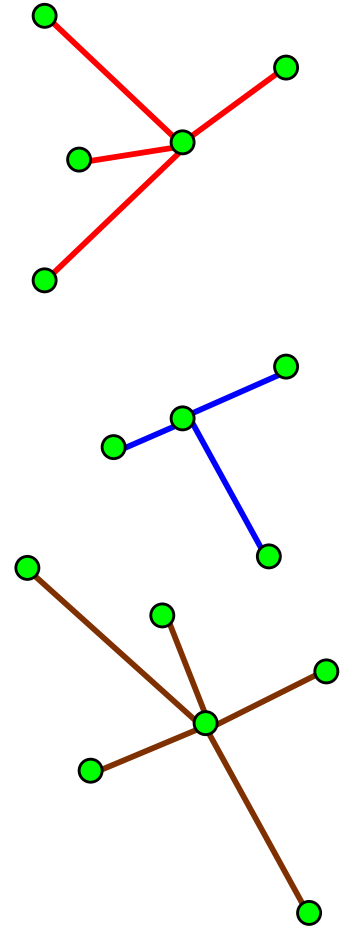
$$\forall i \ : \ T_i \text{ is a star}$$



Star Covers

k-Star Cover: a k-tree cover $\{T_i\}_i$ in which

$$\forall i \ : \ T_i \text{ is a star}$$

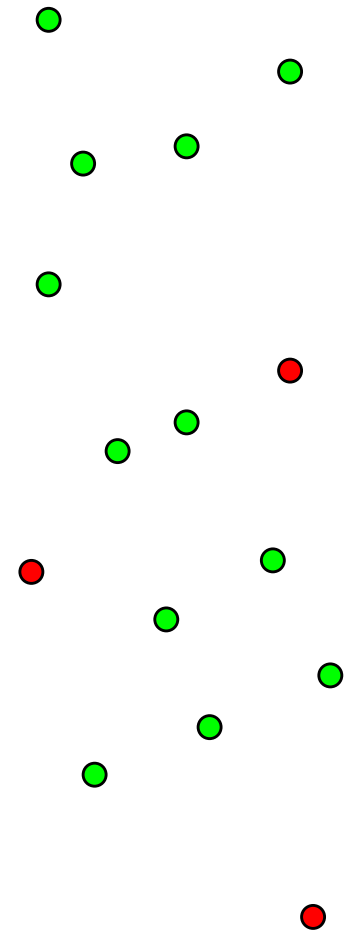


Star Covers

k-Star Cover: a k-tree cover $\{T_i\}_i$ in which

$$\forall i \ : \ T_i \text{ is a star}$$

rooted/unrooted versions: in rooted version, stars must be rooted at R. Namely, r_i is the root of T_i.

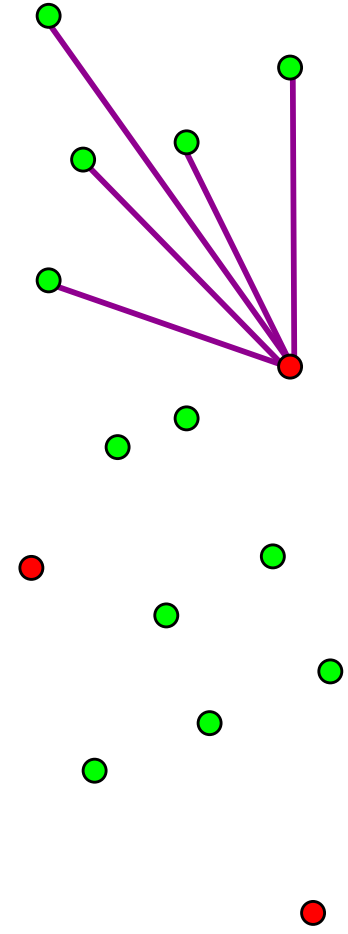


Star Covers

k-Star Cover: a k-tree cover $\{T_i\}_i$ in which

$$\forall i \ : \ T_i \text{ is a star}$$

rooted/unrooted versions: in rooted version, stars must be rooted at R. Namely, r_i is the root of T_i.

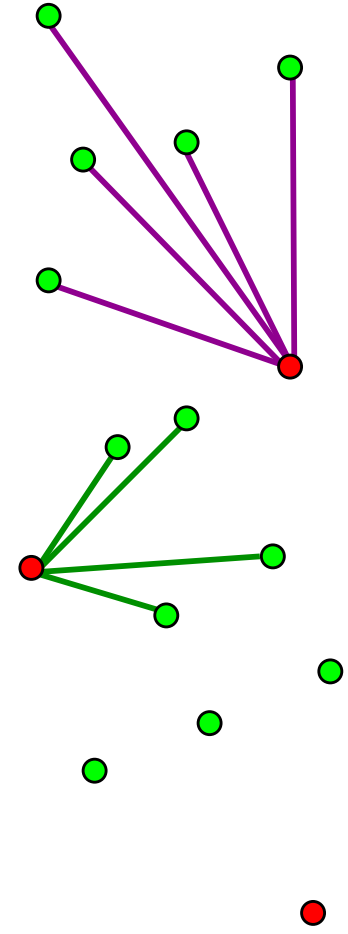


Star Covers

k-Star Cover: a k-tree cover $\{T_i\}_i$ in which

$$\forall i \ : \ T_i \text{ is a star}$$

rooted/unrooted versions: in rooted version, stars must be rooted at R. Namely, r_i is the root of T_i.

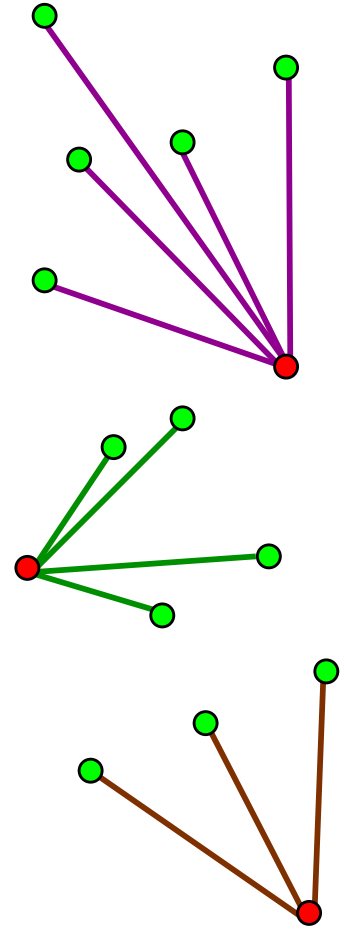


Star Covers

k-Star Cover: a k-tree cover $\{T_i\}_i$ in which

$$\forall i \ : \ T_i \text{ is a star}$$

rooted/unrooted versions: in rooted version, stars must be rooted at R. Namely, r_i is the root of T_i.



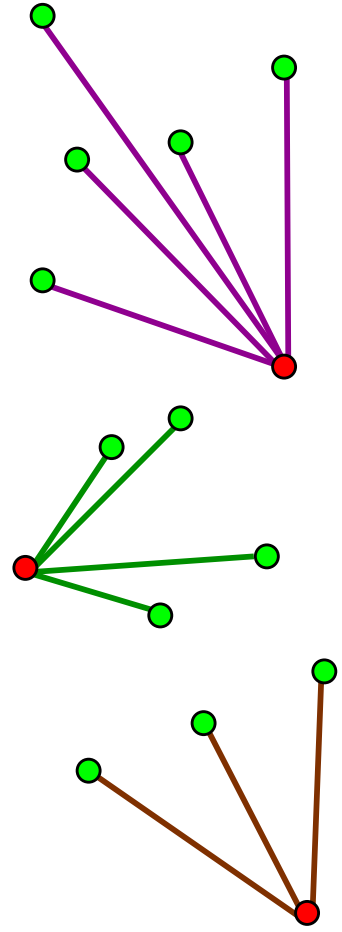
Star Covers

k-Star Cover: a k-tree cover $\{T_i\}_i$ in which

$$\forall i \ : \ T_i \text{ is a star}$$

rooted/unrooted versions: in rooted version, stars must be rooted at R. Namely, r_i is the root of T_i.

motivation : agents must return to base after each visit.



Related work

- k-Traveling Repairman: Cover with tours, $O(1)$-approx minimize average latency. [Fakcharoenphol, Harrelson, Rao 2003]

Related work

- k-Traveling Repairman: Cover with tours, $O(1)$-approx minimize average latency. [Fakcharoenphol, Harrelson, Rao 2003]
- k-Traveling Salesman: Cover with tours, $O(1)$-approx minimize total length. [Haimovich, Rinooy Kan, Stougie 1988]

Related work

- k-Traveling Repairman: Cover with tours, $O(1)$-approx minimize average latency. [Fakcharoenphol, Harrelson, Rao 2003]
- k-Traveling Salesman: Cover with tours, $O(1)$-approx minimize total length. [Haimovich, Rinooy Kan, Stougie 1988]
- Vehicle Routing: Vast amount of work, e.g. Survey [Toth, Vigo, 2002]

Related work - cont

- Chandra Chekuri & Amit Kumar - similar results.

Related work - cont

- Chandra Chekuri & Amit Kumar - similar results.
- Arkin, Hassin, & Levin - approx algorithms for many similar problems:

Related work - cont

- Chandra Chekuri & Amit Kumar - similar results.
- Arkin, Hassin, & Levin - approx algorithms for many similar problems:
  - $O(1)$-approx for unrooted k-path cover.

Related work - cont

- Chandra Chekuri & Amit Kumar - similar results.
- Arkin, Hassin, & Levin - approx algorithms for many similar problems:
  - $O(1)$-approx for unrooted k-path cover.
  - $O(1)$-approx for unrooted B-star cover.

Related work - cont

- Chandra Chekuri & Amit Kumar - similar results.
- Arkin, Hassin, & Levin - approx algorithms for many similar problems:
  - $O(1)$-approx for unrooted k-path cover.
  - $O(1)$-approx for unrooted B-star cover.
  - many other problems...

Results

- Hardness: All 4 problems are NP-complete. (reduction from Bin-Packing).

Results

- Hardness: All 4 problems are NP-complete. (reduction from Bin-Packing).
- k-tree cover: 4-approximation algorithm. Strongly polynomial versions are $(4+\varepsilon)$-approx.

Results

- Hardness: All 4 problems are NP-complete. (reduction from Bin-Packing).
- k-tree cover: 4-approximation algorithm. Strongly polynomial versions are $(4+\varepsilon)$-approx.
- k-star cover:

Results

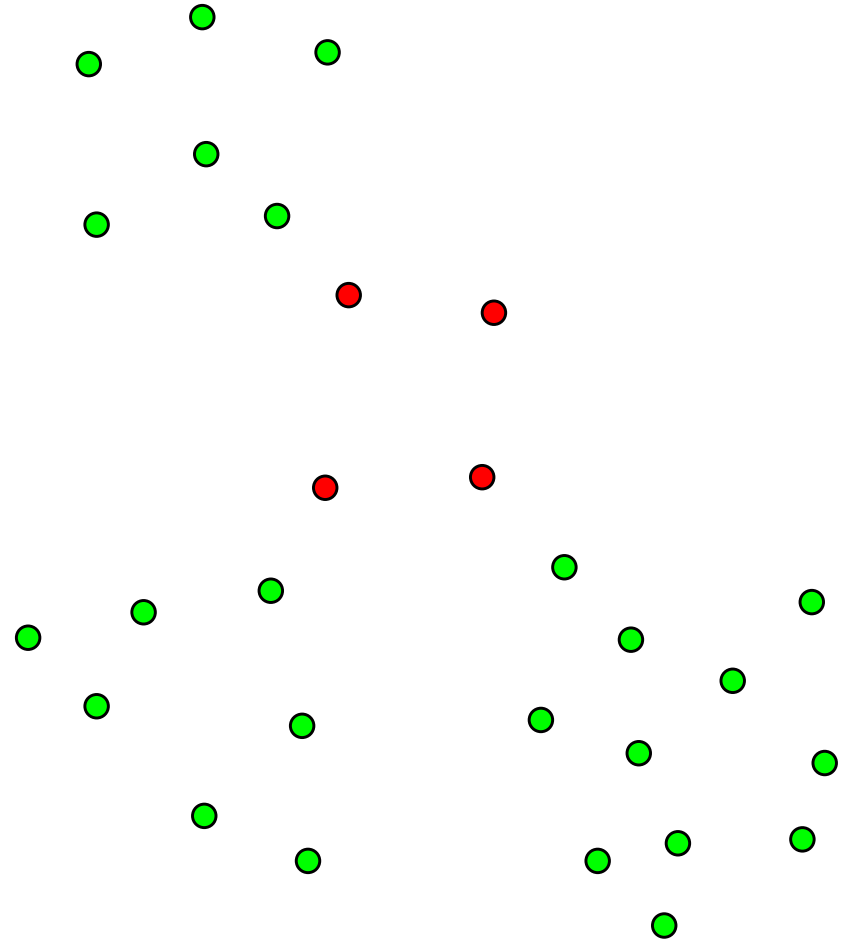
- Hardness: All 4 problems are NP-complete. (reduction from Bin-Packing).
- k-tree cover: 4-approximation algorithm. Strongly polynomial versions are $(4+\varepsilon)$-approx.
- k-star cover:
  - Unrooted version: $(4,4)$-bicriteria approximation algorithm (i.e. $4k$ stars of cost $4 \cdot OPT_k$). Extend method of [Shmoys, Tardos, & Aardal, 1997] for capacitated facility location.

Results

- Hardness: All 4 problems are NP-complete. (reduction from Bin-Packing).
- k-tree cover: 4-approximation algorithm. Strongly polynomial versions are $(4+\varepsilon)$-approx.
- k-star cover:
  - Unrooted version: $(4,4)$-bicriteria approximation algorithm (i.e. $4k$ stars of cost $4 \cdot OPT_k$). Extend method of [Shmoys, Tardos, & Aardal, 1997] for capacitated facility location.
  - Rooted version: equivalent to min. makespan of k machines and n jobs. 2-approximation of [Shmoys & Tardos, 1993].

approximation algorithm: k-rooted tree cover

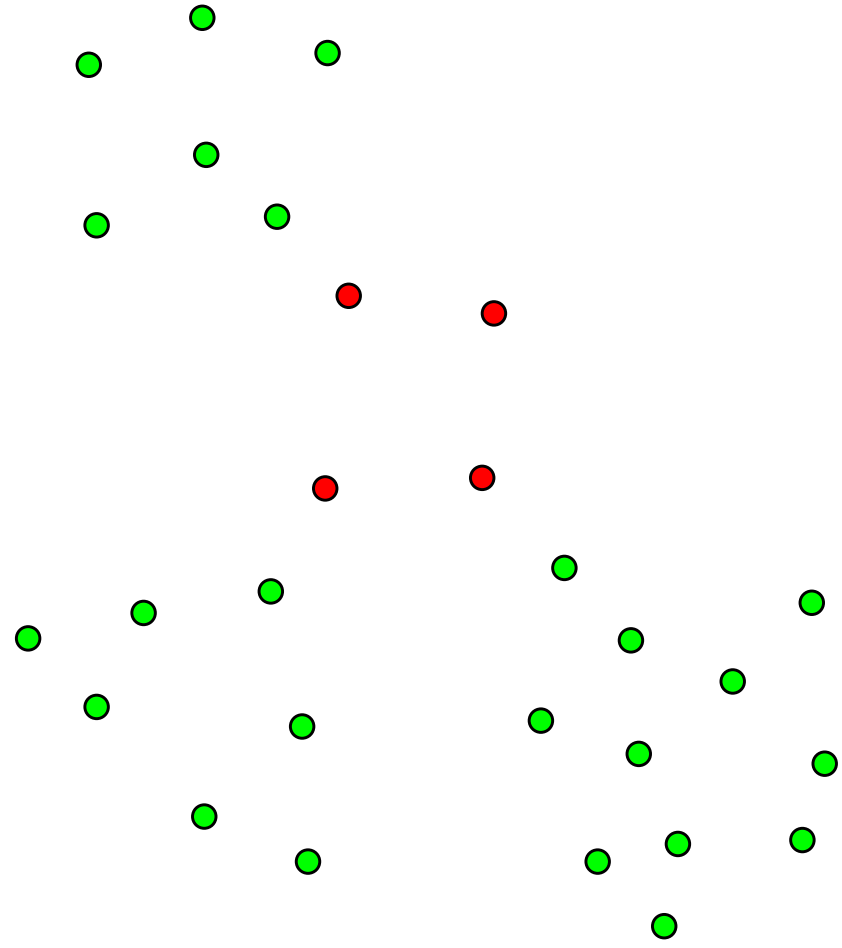
Input: graph, roots, and B - "guess" of opt. cost.



approximation algorithm: k-rooted tree cover

Input: graph, roots, and B - "guess" of opt. cost.

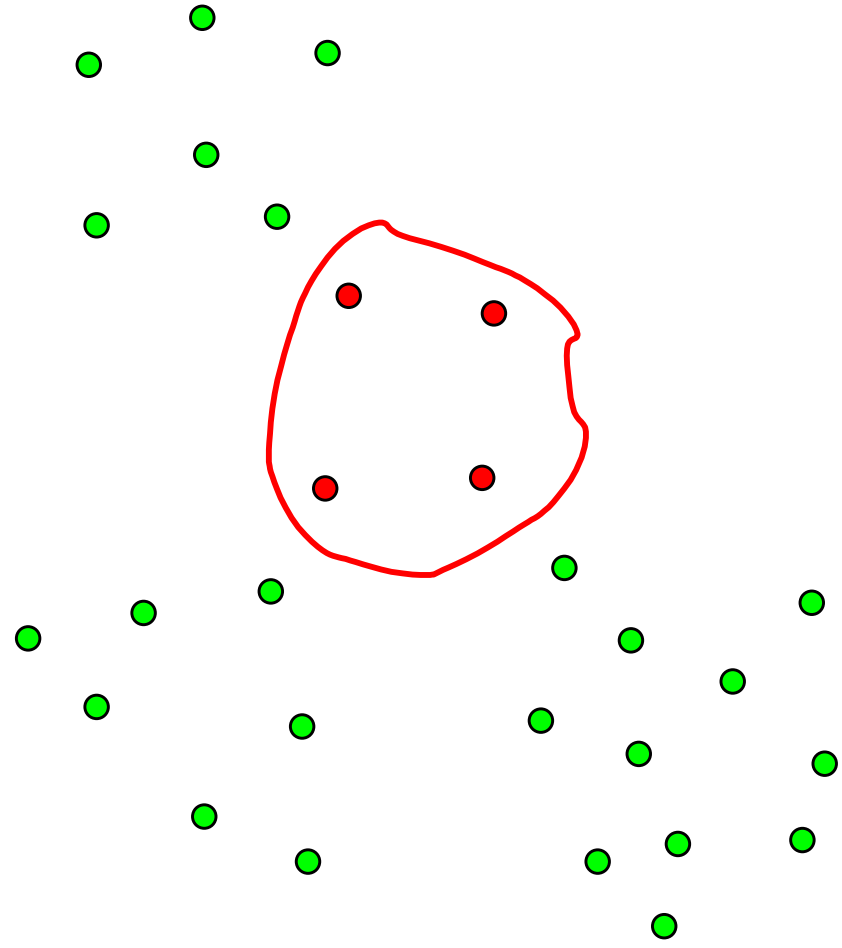
1. contract roots.



approximation algorithm: k-rooted tree cover

Input: graph, roots, and B - "guess" of opt. cost.

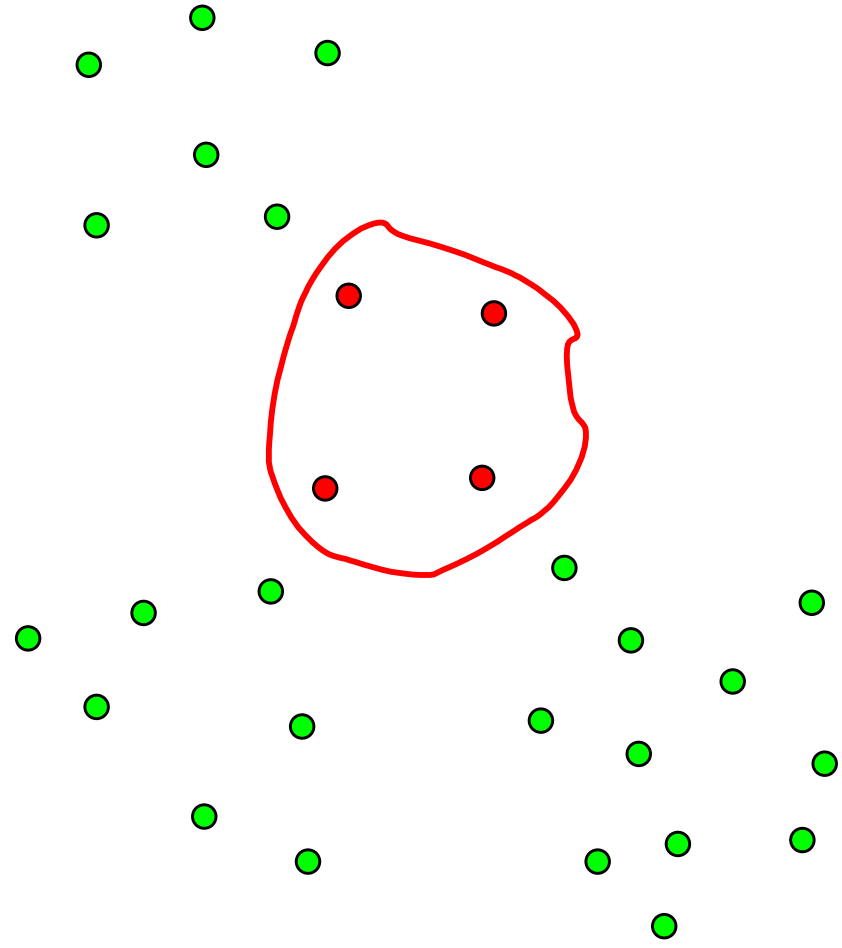
1. contract roots.



approximation algorithm: k-rooted tree cover

Input: graph, roots, and B - "guess" of opt. cost.

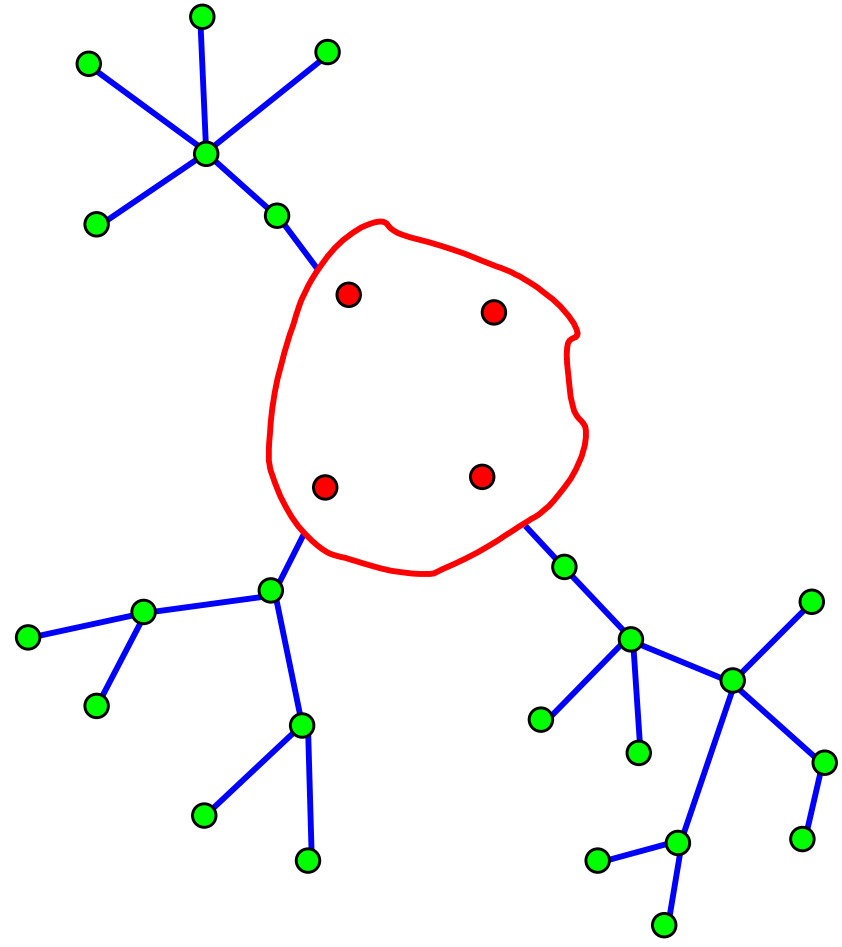
1. contract roots.
2. compute MST.



approximation algorithm: k-rooted tree cover

Input: graph, roots, and B - "guess" of opt. cost.

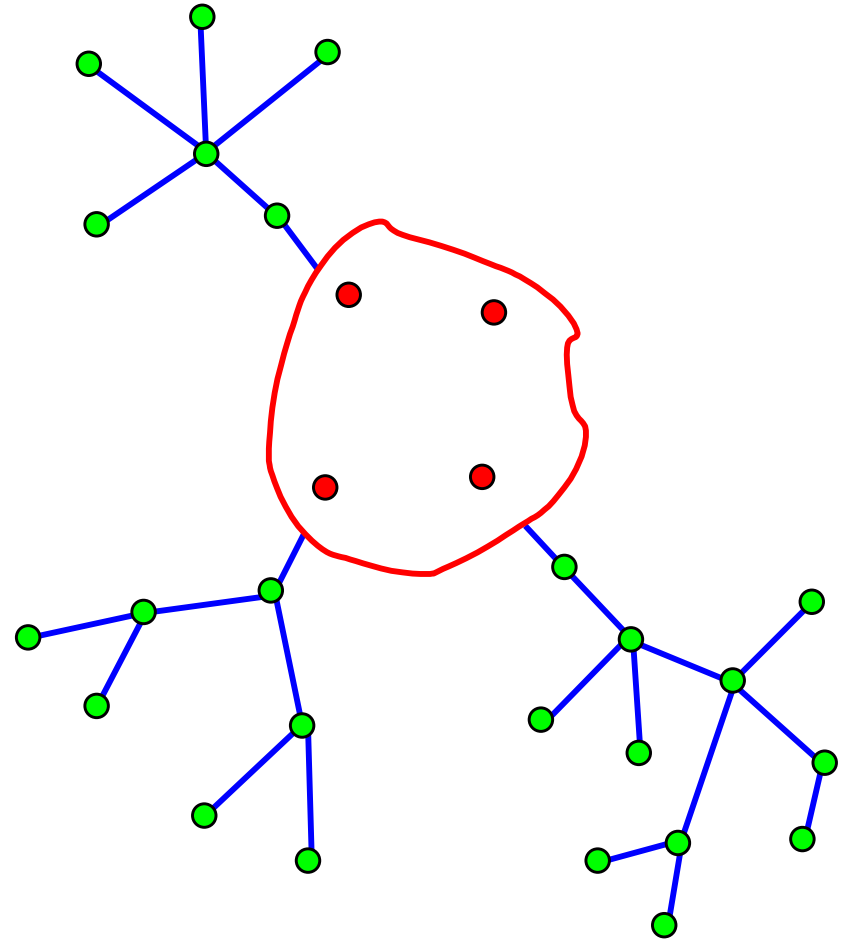
1. contract roots.
2. compute MST.



approximation algorithm: k-rooted tree cover

Input: graph, roots, and B - "guess" of opt. cost.

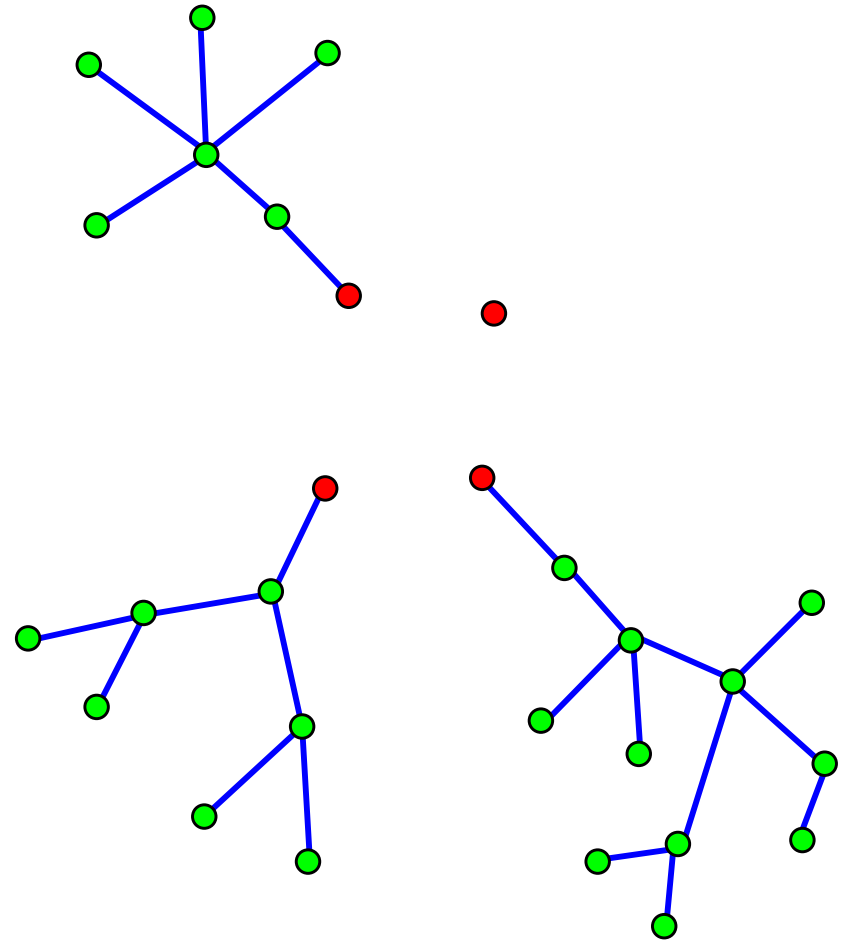
1. contract roots.
2. compute MST.
3. un-contract roots: forest of trees rooted at roots.



approximation algorithm: k-rooted tree cover

Input: graph, roots, and B - "guess" of opt. cost.

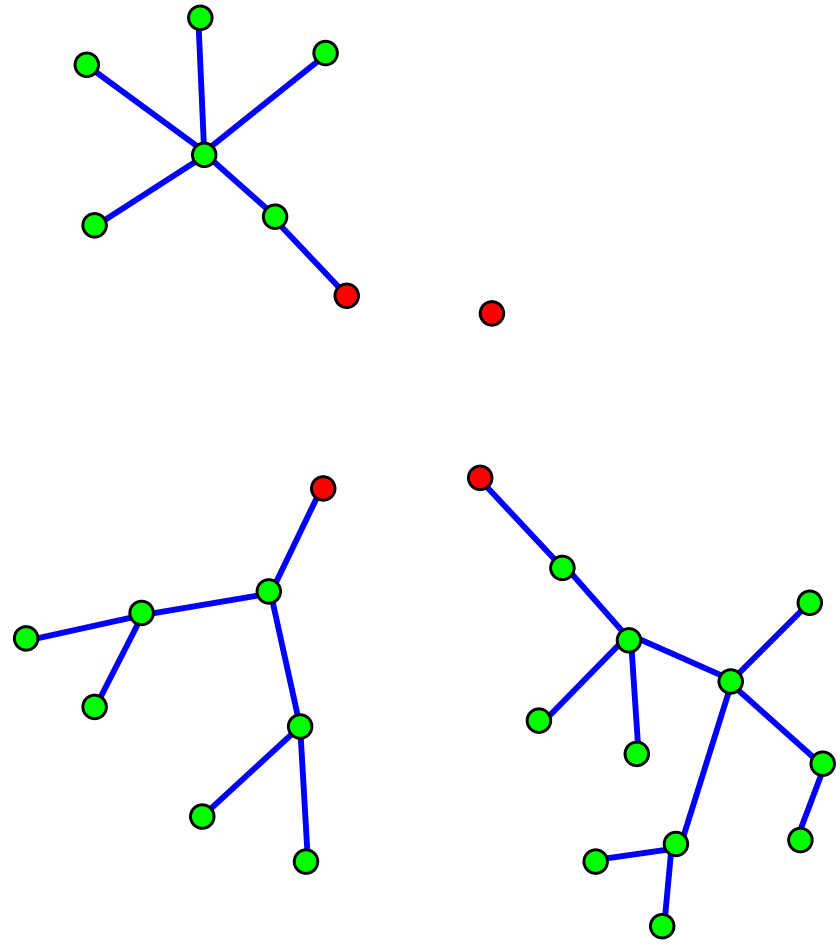
1. contract roots.
2. compute MST.
3. un-contract roots: forest of trees rooted at roots.



approximation algorithm: k-rooted tree cover

Input: graph, roots, and B - "guess" of opt. cost.

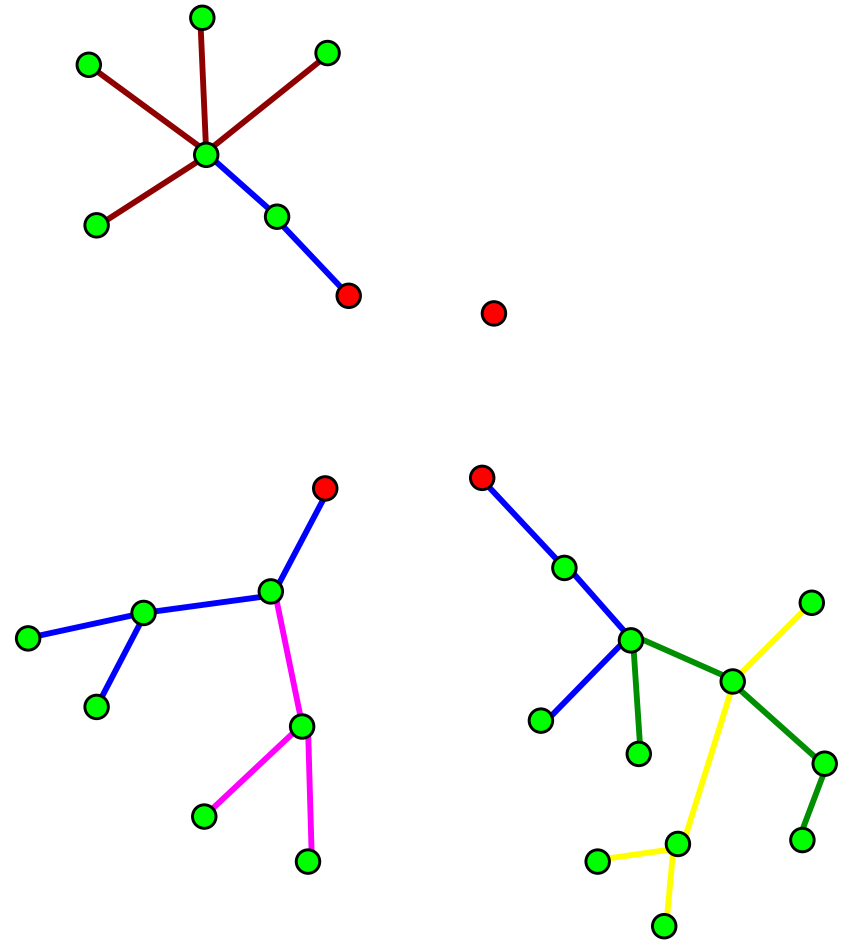
1. contract roots.
2. compute MST.
3. un-contract roots: forest of trees rooted at roots.
4. edge-decompose trees:
   $w(\textit{subtrees}) \in [B, 2B)$,
   $w(\textit{leftovers}) < B$.



approximation algorithm: k-rooted tree cover

Input: graph, roots, and B - "guess" of opt. cost.

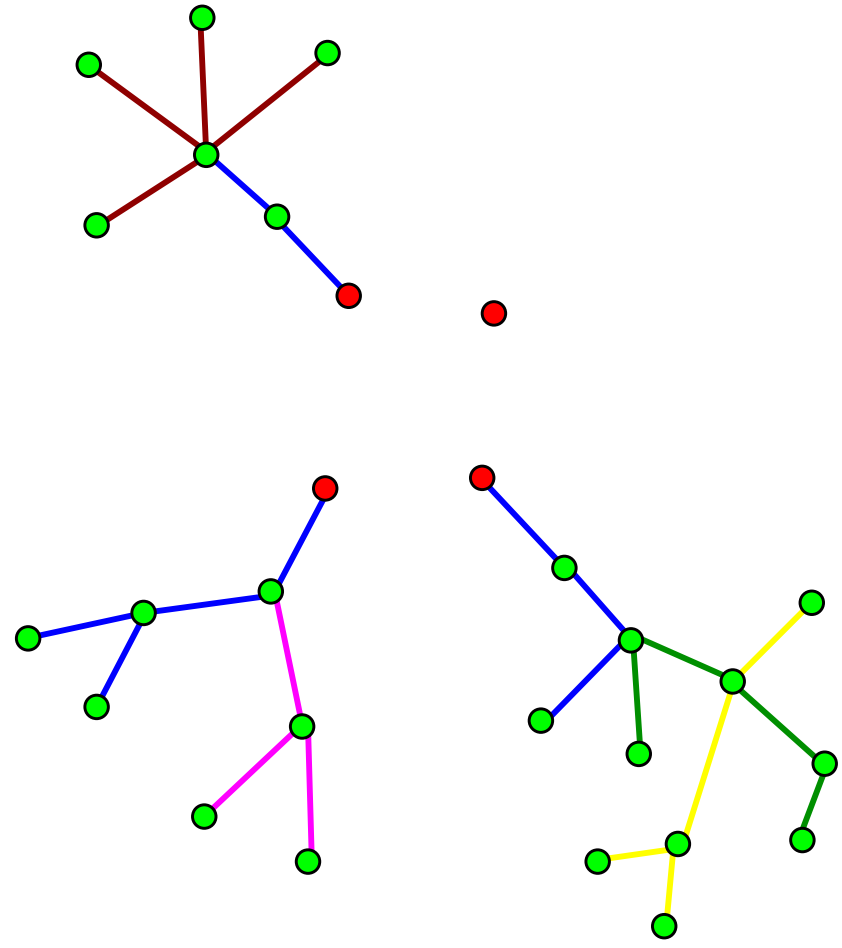
1. contract roots.
2. compute MST.
3. un-contract roots: forest of trees rooted at roots.
4. edge-decompose trees:
   $w(\textit{subtrees}) \in [B, 2B)$,
   $w(\textit{leftovers}) < B$.



approximation algorithm: k-rooted tree cover

Input: graph, roots, and B - "guess" of opt. cost.

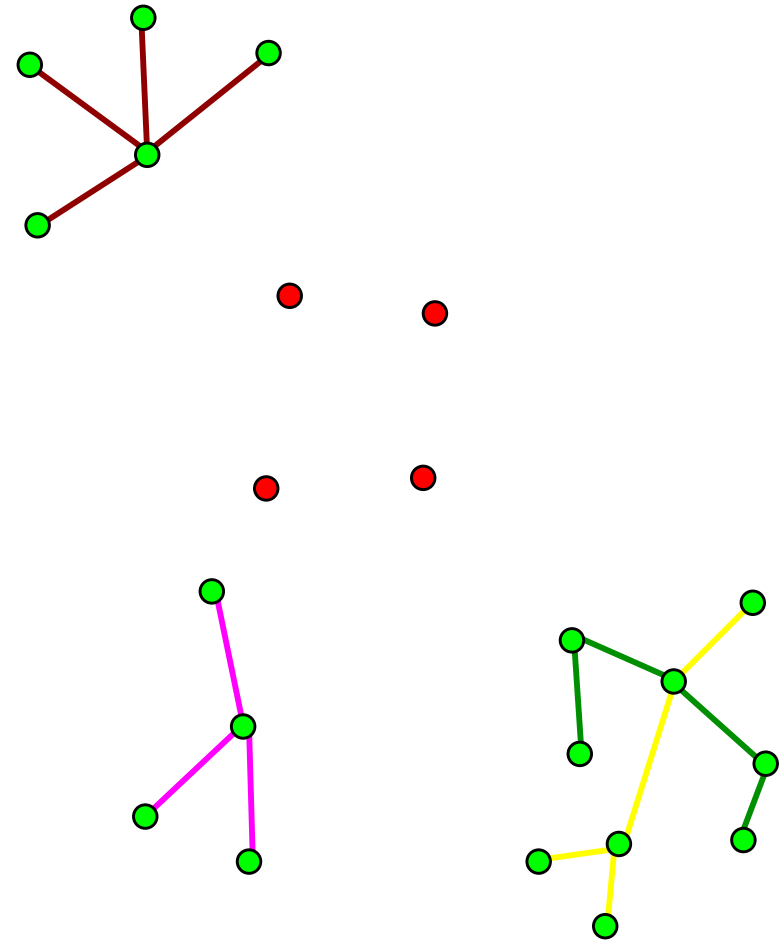
1. contract roots.
2. compute MST.
3. un-contract roots: forest of trees rooted at roots.
4. edge-decompose trees:
   $w(\textit{subtrees}) \in [B, 2B)$,
   $w(\textit{leftovers}) < B$.
5. max match subtrees to roots (if *dist* $\leq B$).



approximation algorithm: k-rooted tree cover

Input: graph, roots, and B - "guess" of opt. cost.

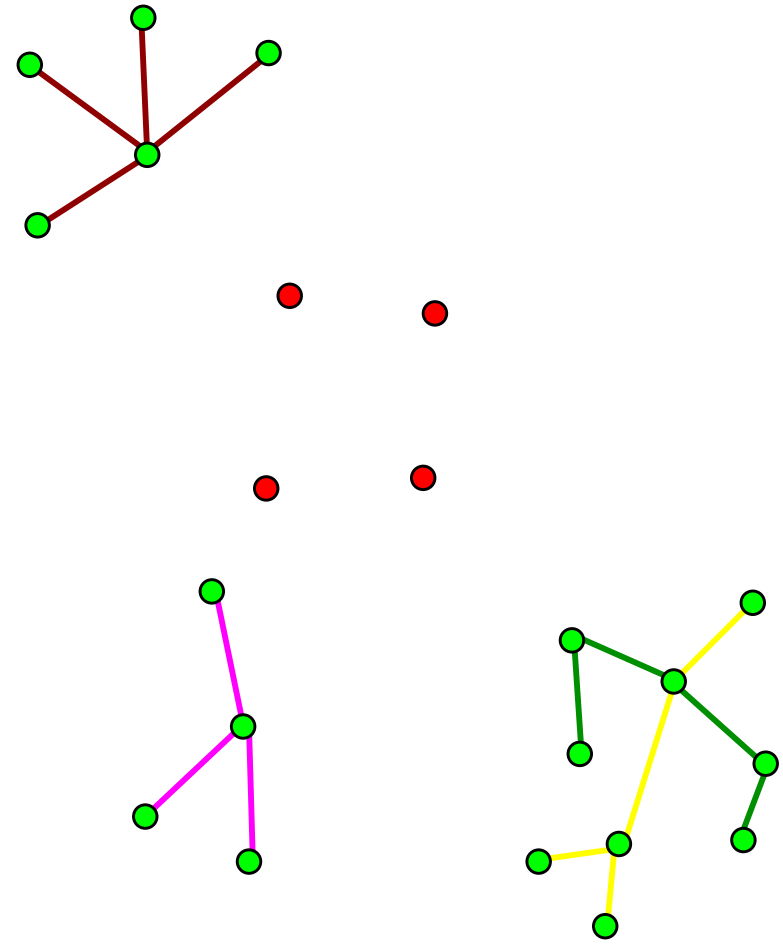
1. contract roots.
2. compute MST.
3. un-contract roots: forest of trees rooted at roots.
4. edge-decompose trees:
   $w(\textit{subtrees}) \in [B, 2B)$,
   $w(\textit{leftovers}) < B$.
5. max match subtrees to roots (if *dist* $\leq B$).



approximation algorithm: k-rooted tree cover

Input: graph, roots, and B - "guess" of opt. cost.

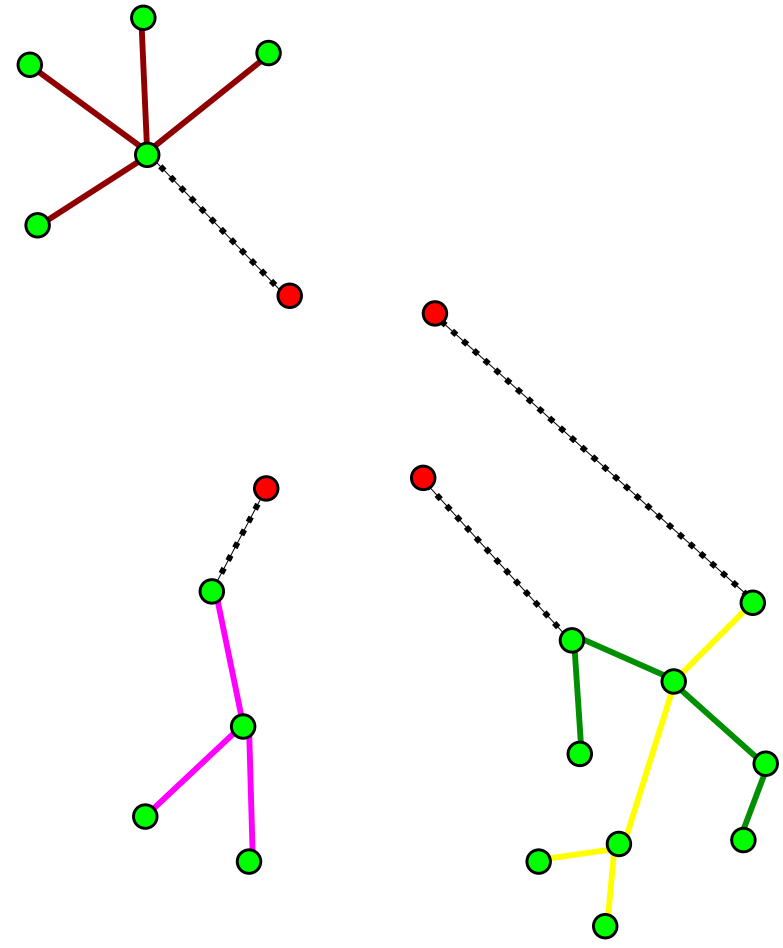
1. contract roots.
2. compute MST.
3. un-contract roots: forest of trees rooted at roots.
4. edge-decompose trees:
   $w(\textit{subtrees}) \in [B, 2B)$,
   $w(\textit{leftovers}) < B$.
5. max match subtrees to roots (if *dist* $\leq B$).
6. if not all subtrees are matched
   $\Rightarrow B < B^*$.



approximation algorithm: k-rooted tree cover

Input: graph, roots, and B - "guess" of opt. cost.

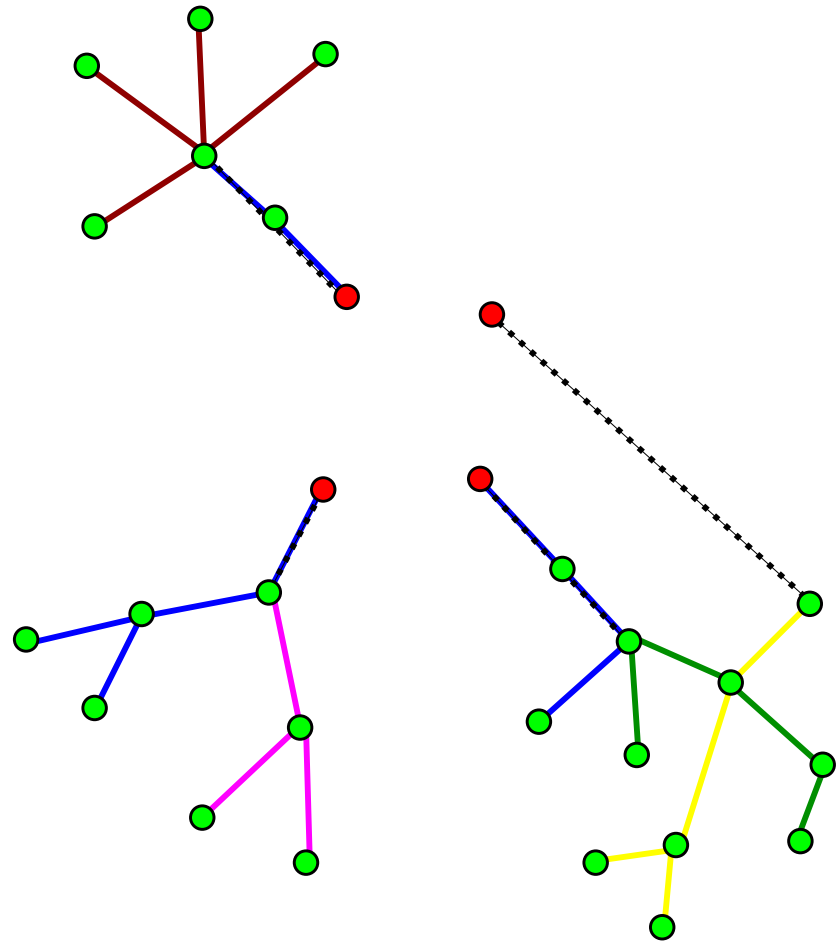
1. contract roots.
2. compute MST.
3. un-contract roots: forest of trees rooted at roots.
4. edge-decompose trees:
   $w(\textit{subtrees}) \in [B, 2B)$,
   $w(\textit{leftovers}) < B$.
5. max match subtrees to roots (if *dist* $\leq B$).
6. if not all subtrees are matched
   $\Rightarrow B < B^*$.



approximation algorithm: k-rooted tree cover

Input: graph, roots, and B - "guess" of opt. cost.

1. contract roots.
2. compute MST.
3. un-contract roots: forest of trees rooted at roots.
4. edge-decompose trees:
   $w(\textit{subtrees}) \in [B, 2B)$,
   $w(\textit{leftovers}) < B$.
5. max match subtrees to roots (if *dist* $\leq B$).
6. if not all subtrees are matched $\Rightarrow B < B^*$.
7. else return $\forall r_i$: leftover + matched subtree.



4-approx algorithm : k-rooted tree cover

- Claim: success $\Rightarrow$ *cost(cover)* $\leq 4 \cdot B$.

4-approx algorithm : k-rooted tree cover

- Claim: success $\Rightarrow$ *cost(cover)* $\leq 4 \cdot B$.
- Claim: fail $\Rightarrow B < B^*$.

4-approx algorithm : k-rooted tree cover

- Claim: success $\Rightarrow$ *cost(cover)* $\leq 4 \cdot B$.
- Claim: fail $\Rightarrow B < B^*$.
- Binary search on value of $B \Rightarrow$ (weakly) polynomial 4-approx algorithm.

4-approx algorithm : k-rooted tree cover

- Claim: success $\Rightarrow$ *cost(cover)* $\leq 4 \cdot B$.
- Claim: fail $\Rightarrow B < B^*$.
- Binary search on value of $B \Rightarrow$ (weakly) polynomial 4-approx algorithm.
- Scaling $\Rightarrow$ strongly polynomial $(4 + \varepsilon)$-approx algorithm.

Claim: success $\Rightarrow$ *cost(cover)* $\leq 4 \cdot B$

Claim: success $\Rightarrow$ *cost(cover)* $\leq 4 \cdot B$

Each tree in tree cover may consist of:

- a rooted leftover subtree $\Rightarrow$ *cost(leftover)* $< B$.

Claim: success $\Rightarrow$ *cost(cover)* $\leq 4 \cdot B$

Each tree in tree cover may consist of:

- a rooted leftover subtree $\Rightarrow$ *cost(leftover)* $< B$.
- a matched subtree $\Rightarrow$ *cost(subtree)* $< 2 \cdot B$.

Claim: success $\Rightarrow$ *cost(cover)* $\leq 4 \cdot B$

Each tree in tree cover may consist of:

- a rooted leftover subtree $\Rightarrow$ *cost(leftover)* $< B$.
- a matched subtree $\Rightarrow$ *cost(subtree)* $< 2 \cdot B$.
- matching edge $\Rightarrow$ *cost(edge)* $\leq B$.

Claim: success $\Rightarrow$ *cost(cover)* $\leq 4 \cdot B$

Each tree in tree cover may consist of:

- a rooted leftover subtree $\Rightarrow$ *cost(leftover)* $< B$.
- a matched subtree $\Rightarrow$ *cost(subtree)* $< 2 \cdot B$.
- matching edge $\Rightarrow$ *cost(edge)* $\leq B$.

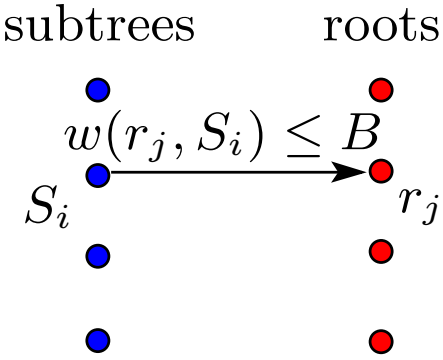
$\Rightarrow$ weight of every tree in solution is $< 4 \cdot B$. QED

Claim: fail $\Rightarrow B < B^*$

Claim: fail $\Rightarrow B < B^*$

Assume for sake of contradiction:

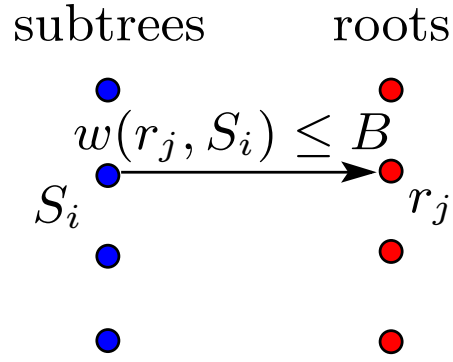
$B \geq B^*$ and matching failed.



Claim: fail $\Rightarrow B < B^*$

Assume for sake of contradiction:

$B \geq B^*$ and matching failed.



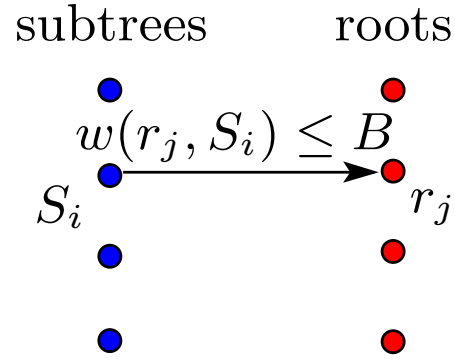
Hall's Theorem: if matching failed, then

$\exists \mathcal{S} \subseteq$ subtrees : $|\mathcal{S}| > |N(\mathcal{S})|$.

Claim: fail $\Rightarrow B < B^*$

Assume for sake of contradiction:

$B \geq B^*$ and matching failed.



Hall's Theorem: if matching failed, then

$\exists \mathcal{S} \subseteq$ subtrees : $|\mathcal{S}| > |N(\mathcal{S})|$.

Fix OPT:

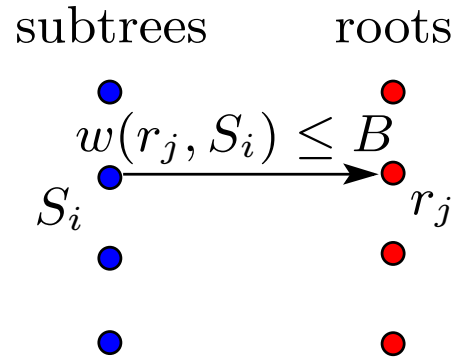
$$\mathcal{T}^* \triangleq \{T_1^*, \ldots, T_k^*\} \text{ where } r_j \in T_j^*.$$

$$\mathcal{T}^*(\mathcal{S}) \triangleq \{T_j^* \mid \exists S_i \in \mathcal{S} : T_j^* \cap S_i \neq \emptyset\}.$$

Claim: fail $\Rightarrow B < B^*$

Assume for sake of contradiction:

$B \geq B^*$ and matching failed.



Hall's Theorem: if matching failed, then

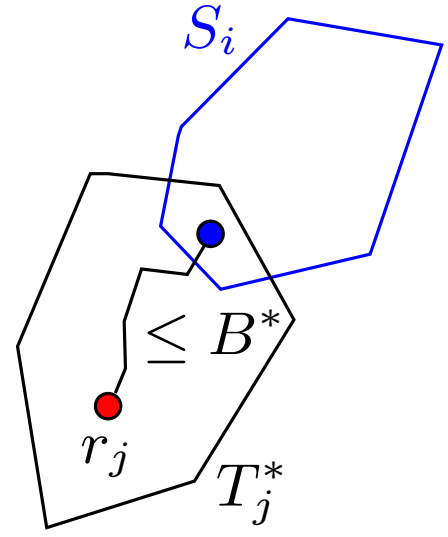
$\exists \mathcal{S} \subseteq$ subtrees : $|\mathcal{S}| > |N(\mathcal{S})|$.

Fix OPT:

$\mathcal{T}^* \triangleq \{T_1^*, \ldots, T_k^*\}$ where $r_j \in T_j^*$.

$\mathcal{T}^*(\mathcal{S}) \triangleq \{T_j^* \mid \exists S_i \in \mathcal{S} : T_j^* \cap S_i \neq \emptyset\}$.

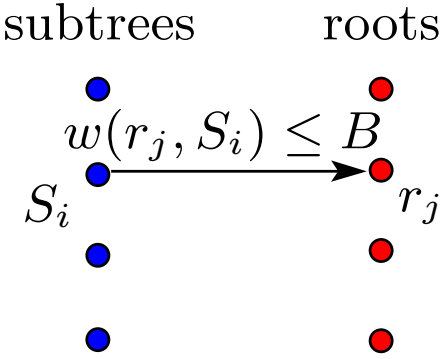
Note: $T_j^* \cap S_i \neq \emptyset \quad \Rightarrow \quad w(r_j, S_i) \leq B^* \leq B$.



Claim: fail $\Rightarrow B < B^*$

Assume for sake of contradiction:

$B \geq B^*$ and matching failed.



Hall's Theorem: if matching failed, then

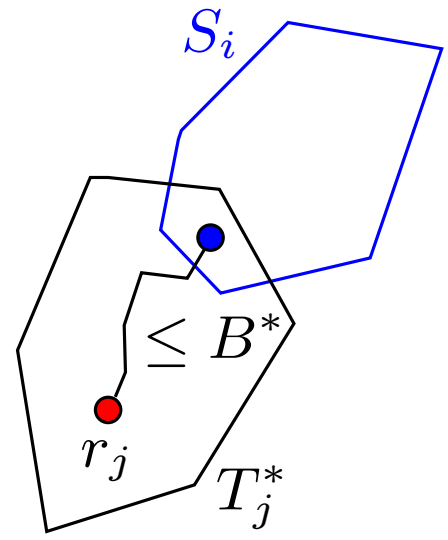
$\exists \mathcal{S} \subseteq$ subtrees : $|\mathcal{S}| > |N(\mathcal{S})|$.

Fix OPT:

$\mathcal{T}^* \triangleq \{T_1^*, \ldots, T_k^*\}$ where $r_j \in T_j^*$.

$\mathcal{T}^*(\mathcal{S}) \triangleq \{T_j^* \mid \exists S_i \in \mathcal{S} : T_j^* \cap S_i \neq \emptyset\}$.

Note: $T_j^* \cap S_i \neq \emptyset \quad \Rightarrow \quad w(r_j, S_i) \leq B^* \leq B$.



$\Rightarrow |\mathcal{T}^*(\mathcal{S})| \leq |N(\mathcal{S})|$.

Claim: fail $\Rightarrow B < B^*$ - (cont.)

recall:

$$B \geq B^* \quad \text{and} \quad |\mathcal{T}^*(\mathcal{S})| \leq |N(\mathcal{S})| < |\mathcal{S}|.$$

Claim: fail $\Rightarrow B < B^*$ - (cont.)

recall:

$$B \geq B^* \quad \text{and} \quad |\mathcal{T}^*(\mathcal{S})| \leq |N(\mathcal{S})| < |\mathcal{S}|.$$

$$\begin{aligned} \forall j : w(T_j^*) \leq B^* &\Rightarrow w(\mathcal{T}^*(\mathcal{S})) \leq B^* \cdot |\mathcal{T}^*(\mathcal{S})| \\ \forall i : w(S_i) \in [B, 2B) &\Rightarrow w(\mathcal{S}) \geq B \cdot |\mathcal{S}|. \\ &\Rightarrow w(\mathcal{S}) > w(\mathcal{T}^*(\mathcal{S})). \end{aligned}$$

Claim: fail $\Rightarrow B < B^*$ - (cont.)

recall:

$$B \geq B^* \quad \text{and} \quad |\mathcal{T}^*(\mathcal{S})| \leq |N(\mathcal{S})| < |\mathcal{S}|.$$

$$\begin{aligned} \forall j : w(T_j^*) \leq B^* &\Rightarrow w(\mathcal{T}^*(\mathcal{S})) \leq B^* \cdot |\mathcal{T}^*(\mathcal{S})| \\ \forall i : w(S_i) \in [B, 2B) &\Rightarrow w(\mathcal{S}) \geq B \cdot |\mathcal{S}|. \\ &\Rightarrow w(\mathcal{S}) > w(\mathcal{T}^*(\mathcal{S})). \end{aligned}$$

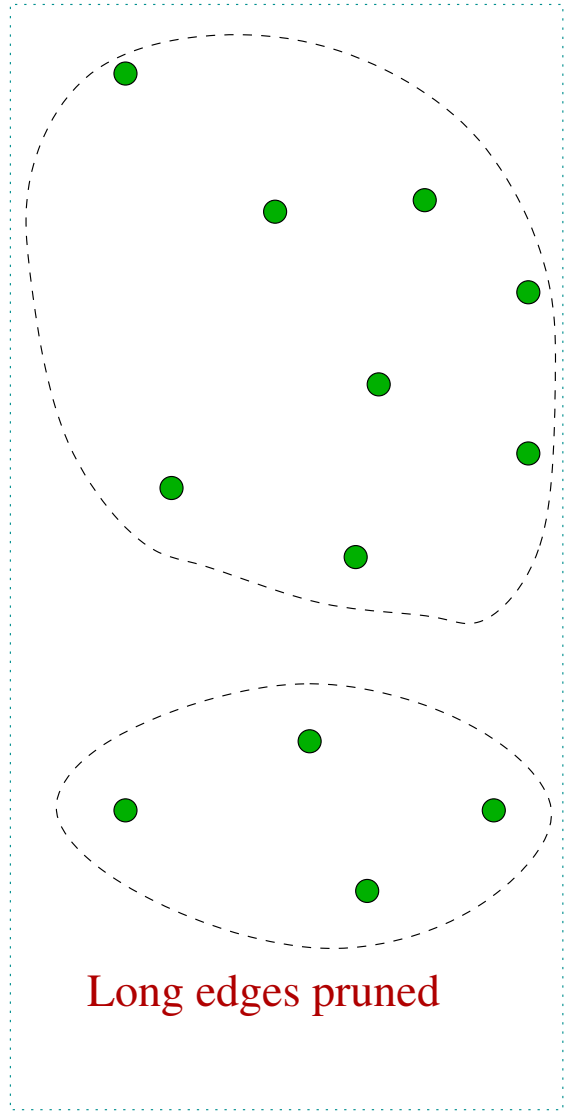
But

$$T' \triangleq MST + \mathcal{T}^*(\mathcal{S}) - \mathcal{S}$$

is a spanning tree and $w(T') < w(MST)$, contradiction. QED

Algorithm for Unrooted k-tree cover

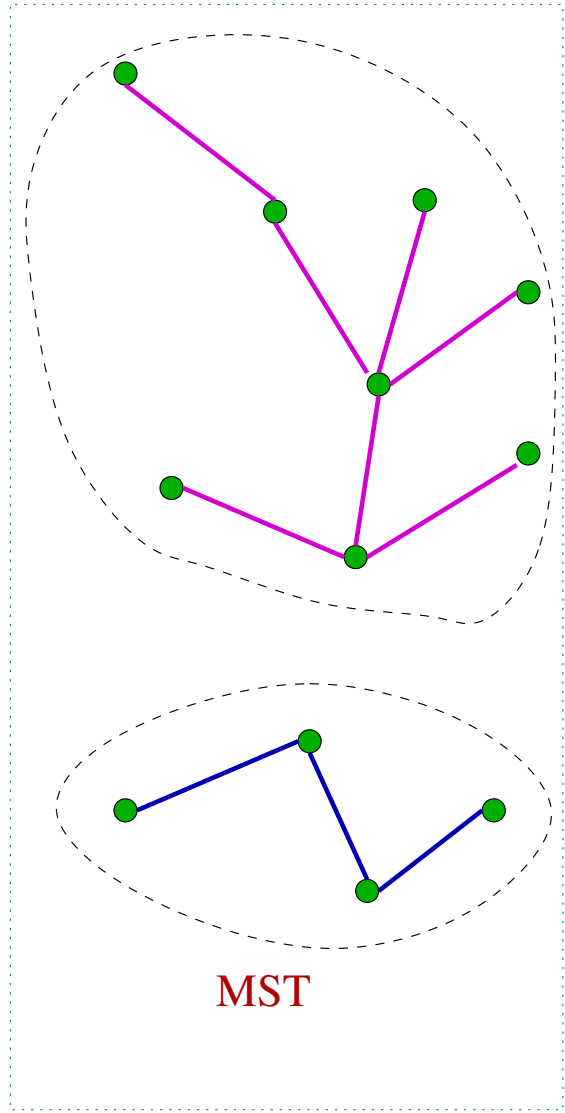
1. Prune edges $w_e > B$.
   Let $\{G_i\}_i$ be components.



Algorithm for Unrooted k-tree cover

1. Prune edges $w_e > B$.
   Let $\{G_i\}_i$ be components.

2. MST_i = MST of G_i.
   $k_i = \lfloor \frac{w(MST_i)}{2B} \rfloor$.

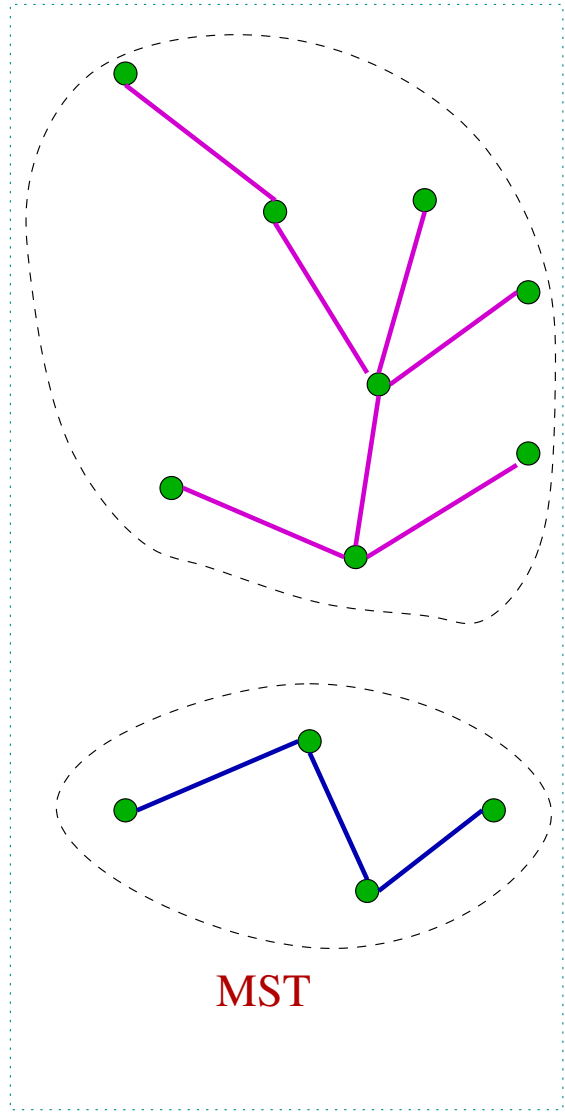


Algorithm for Unrooted k-tree cover

1. Prune edges $w_e > B$.
   Let $\{G_i\}_i$ be components.

2. MST_i = MST of G_i.
   $k_i = \lfloor \frac{w(MST_i)}{2B} \rfloor$.

3. If $\sum_i (k_i + 1) > k$, return "fail".



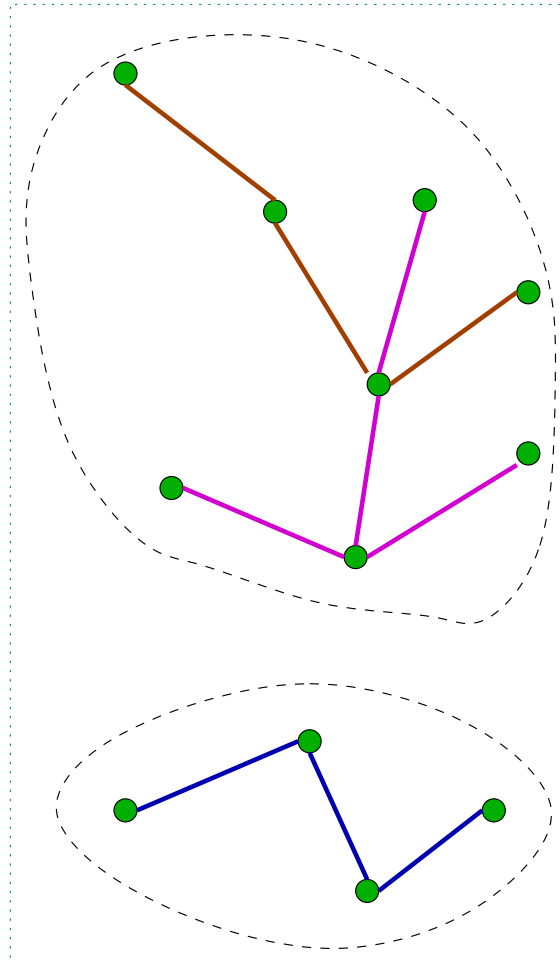
Algorithm for Unrooted k-tree cover

1. Prune edges $w_e > B$.
   Let $\{G_i\}_i$ be components.

2. MST_i = MST of G_i.
   $k_i = \lfloor \frac{w(MST_i)}{2B} \rfloor$.

3. If $\sum_i (k_i + 1) > k$, return "fail".

4. Decompose each MST_i into at most k_i+1 trees $S_i^1 + \ldots + S_i^{k_i} + L_i$ such that $w(S_i^j) \in [2B, 4B)$ and $w(L_i) < 2B$. Return "success".



Final solution

Analysis

Claim: On success, each tree has weight no more than $4B$.

Analysis

Claim: On success, each tree has weight no more than $4B$.

Claim: On failure, $B < B^*$.

Analysis

Claim: On success, each tree has weight no more than $4B$.

Claim: On failure, $B < B^*$.

Alternatively, if $B^* \leq B$, then $k_i + 1 \leq k_i^*$ for all i.

Proof: $B^* \leq B \Rightarrow k_i + 1 \leq k_i^*$

Proof: $B^* \leq B \Rightarrow k_i + 1 \leq k_i^*$

Let optimal solution cover G_i with $\{T_1^*, \ldots, T_{k_i^*}^*\}$.

Proof: $B^* \leq B \Rightarrow k_i + 1 \leq k_i^*$

Let optimal solution cover G_i with $\{T_1^*, \ldots, T_{k_i^*}^*\}$.
Augment it to span G_i by adding $k_i^* - 1$ edges, so:

$$\sum_{j=1}^{k_i^*} w(T_i^*) + (k_i^* - 1)B \geq w(MST_i)$$

Proof: $B^* \leq B \Rightarrow k_i + 1 \leq k_i^*$

Let optimal solution cover G_i with $\{T_1^*, \ldots, T_{k_i^*}^*\}$.
Augment it to span G_i by adding $k_i^* - 1$ edges, so:

$$\sum_{j=1}^{k_i^*} w(T_i^*) + (k_i^* - 1)B \geq w(MST_i)$$

Since $w(T_i^*) \leq B^* \leq B$,

$$k_i^* \cdot B + (k_i^* - 1)B \geq w(MST_i).$$

Proof: $B^* \leq B \Rightarrow k_i + 1 \leq k_i^*$

Let optimal solution cover G_i with $\{T_1^*, \ldots, T_{k_i^*}^*\}$.
Augment it to span G_i by adding $k_i^* - 1$ edges, so:

$$\sum_{j=1}^{k_i^*} w(T_i^*) + (k_i^* - 1)B \geq w(MST_i)$$

Since $w(T_i^*) \leq B^* \leq B$,

$$k_i^* \cdot B + (k_i^* - 1)B \geq w(MST_i).$$

$$\Rightarrow k_i^* \geq \frac{w(MST_i)}{2B} + \frac{1}{2} > k_i.$$

□