Covering Graphs Using Trees and Stars

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and

Jochen Könemann, R. Ravi and A. Sinha (Pittsburgh)

Third Haifa Workshop on Interdisciplinary Applications of Graph Theory, Combinatorics and Computing

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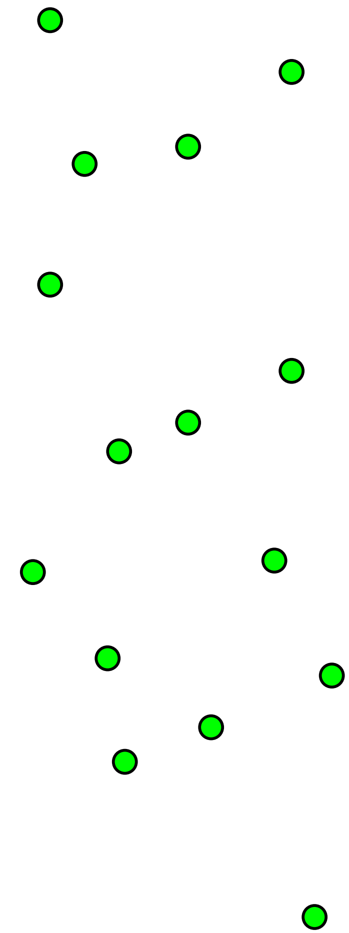

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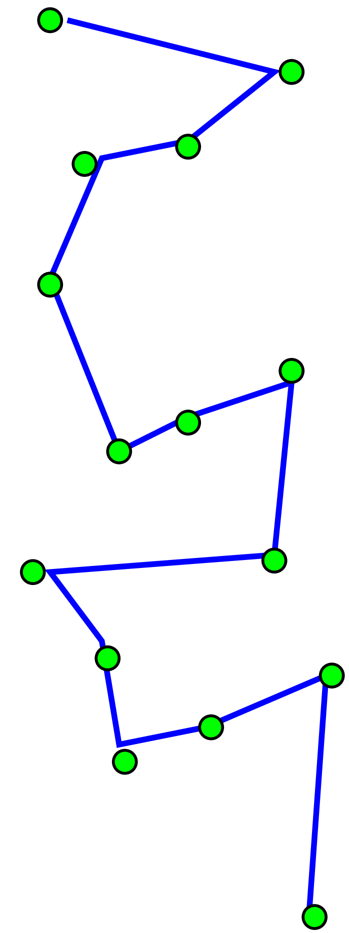

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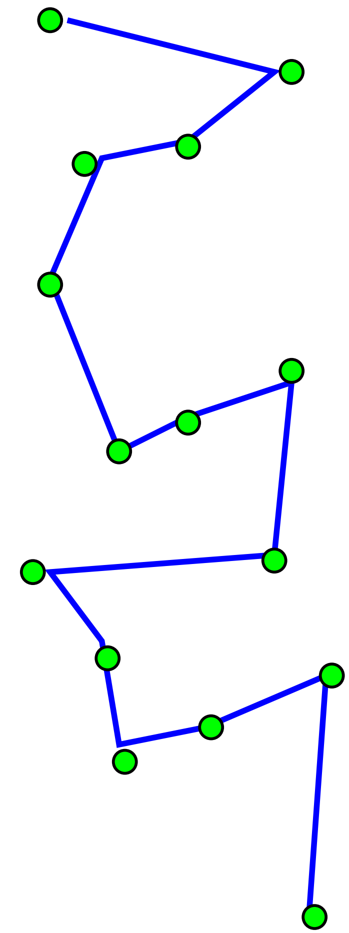

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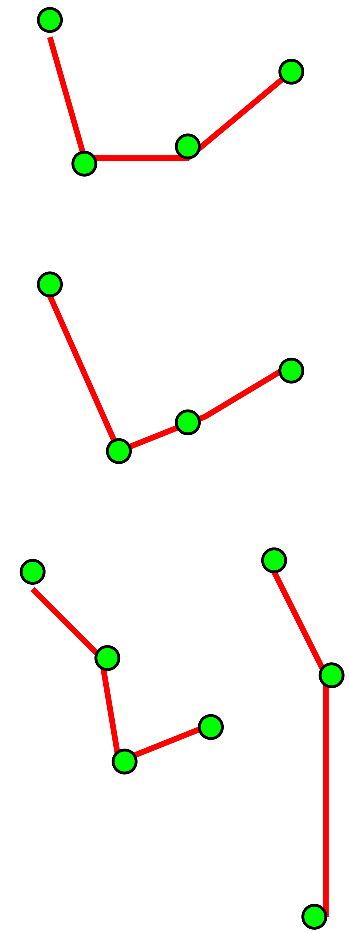

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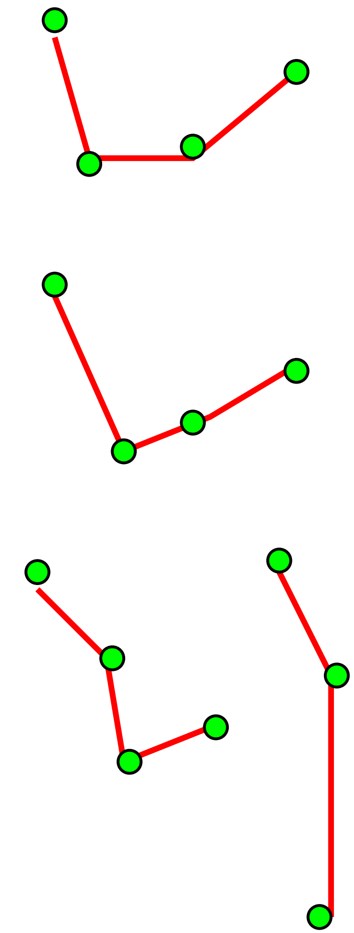

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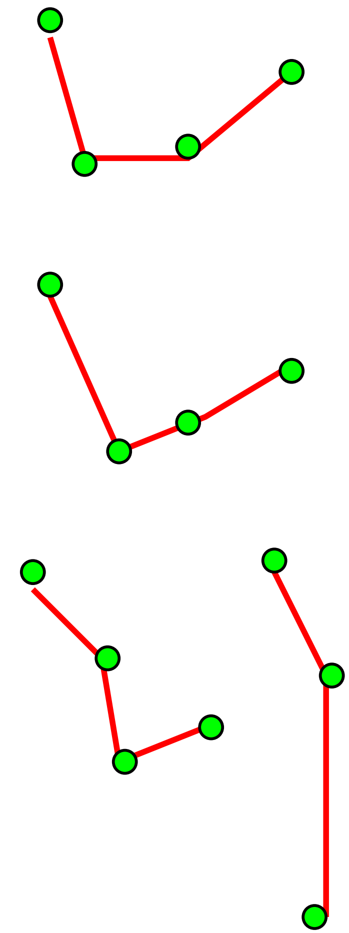

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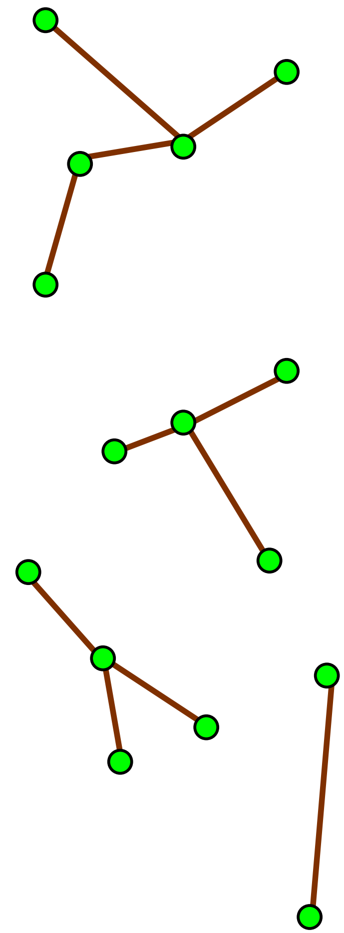
- MST is a constant ratio approx of a min tour $\Rightarrow$ k-Tree Cover Problem.



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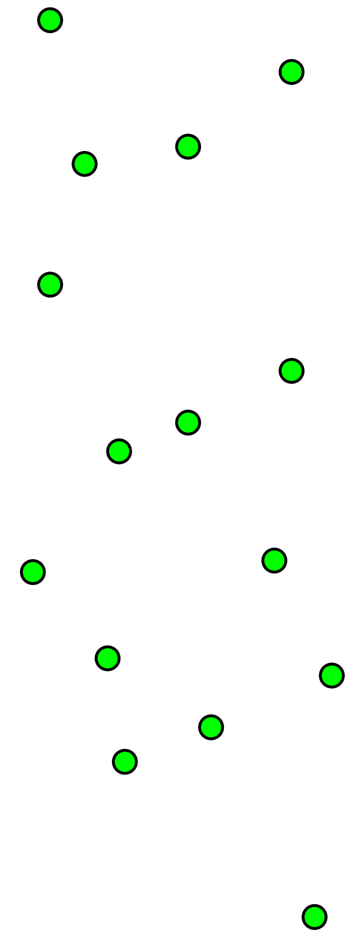

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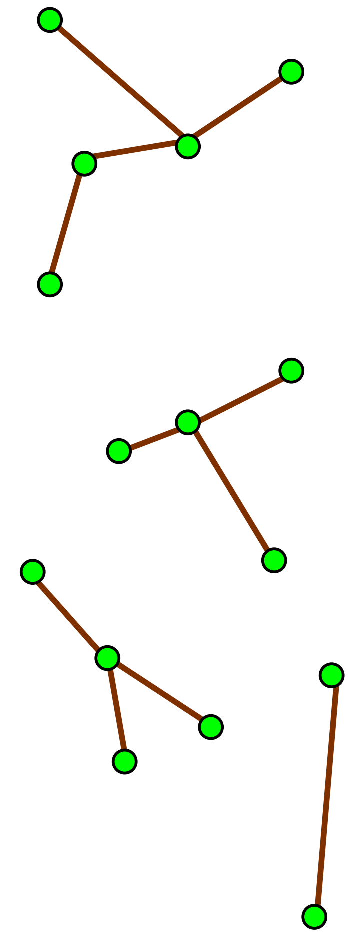

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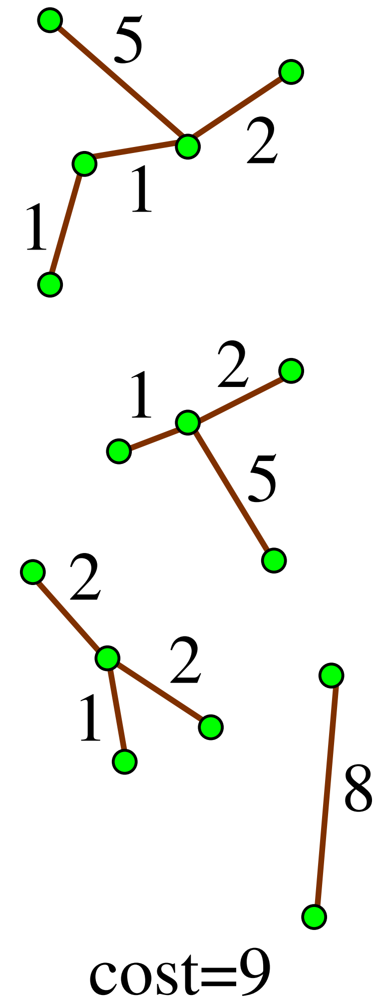

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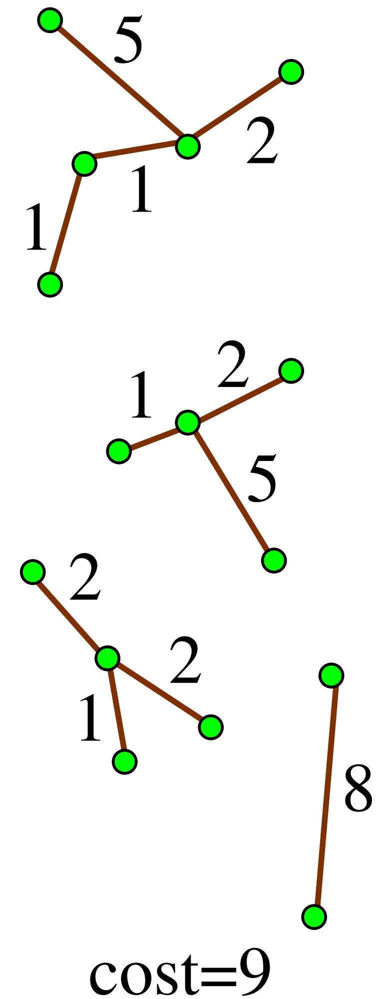

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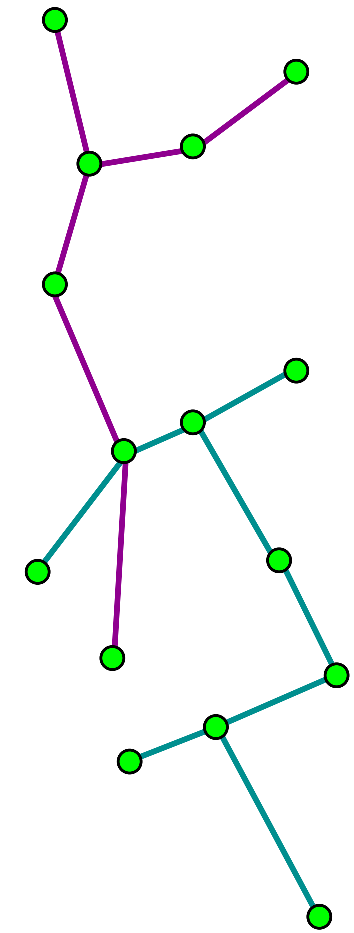
remark : trees may share nodes & edges in a tree cover.



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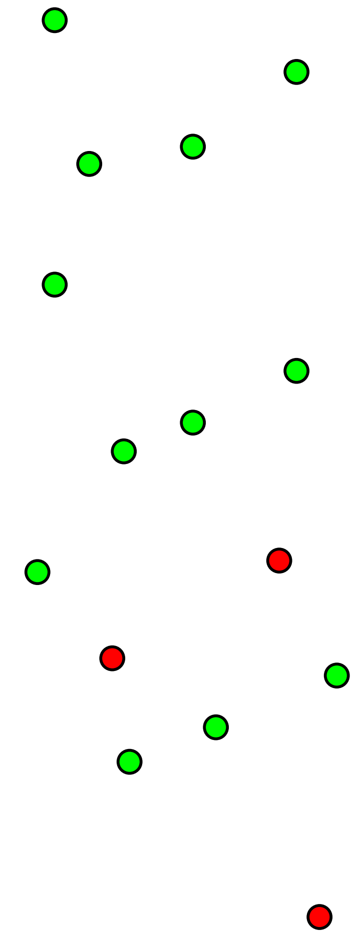

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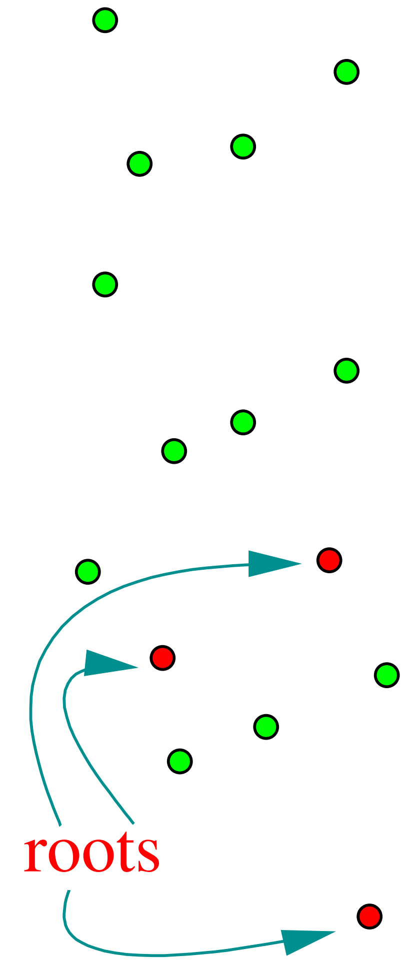

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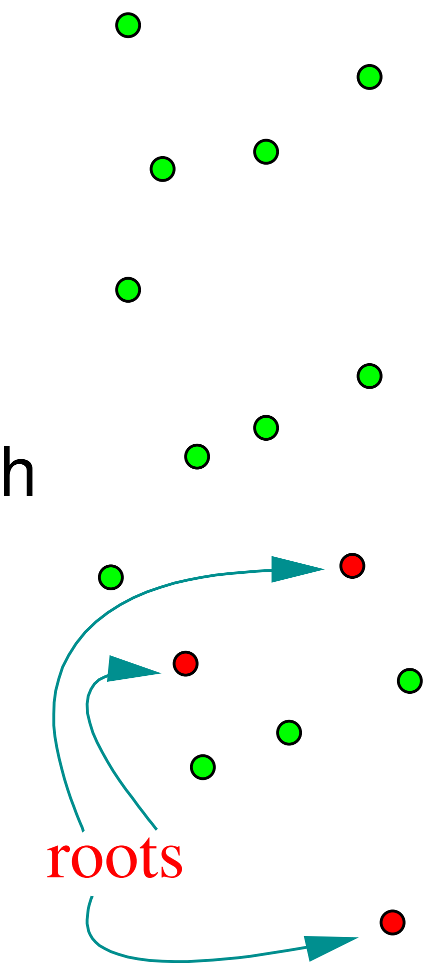

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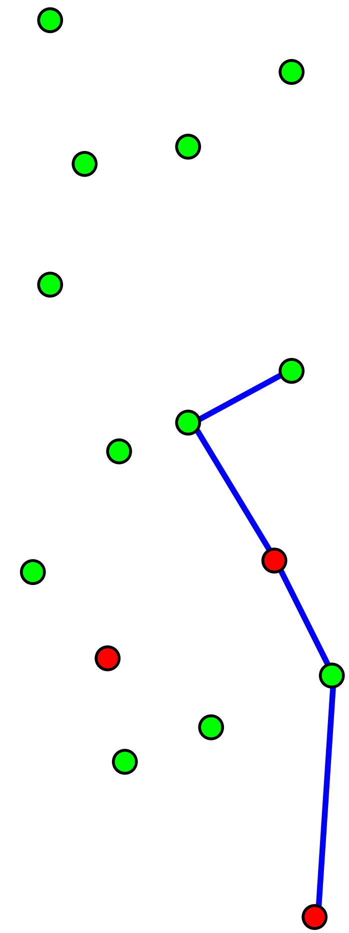

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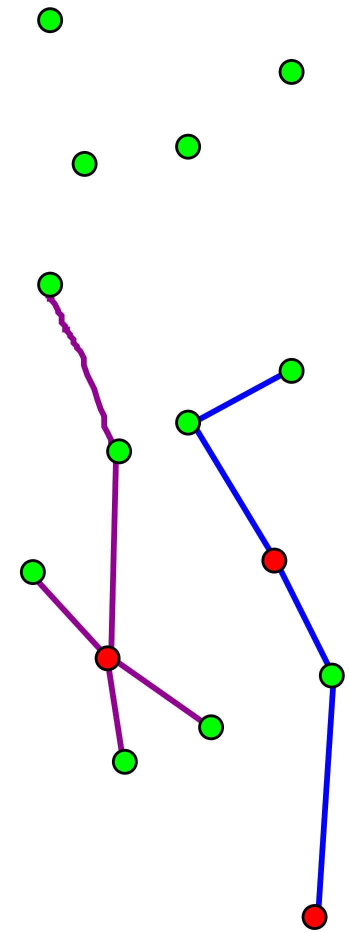

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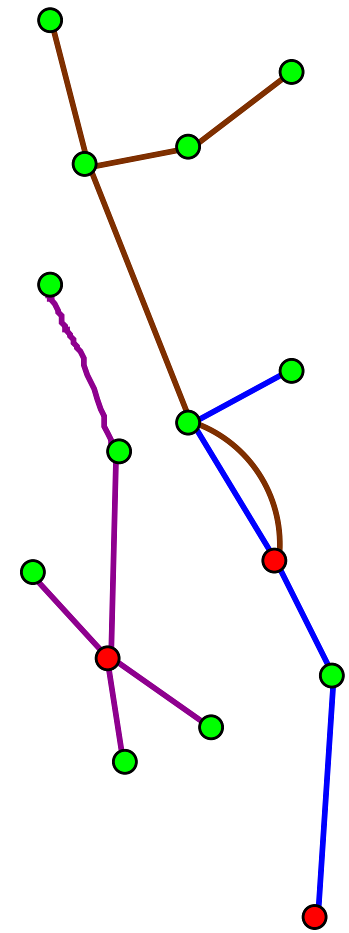

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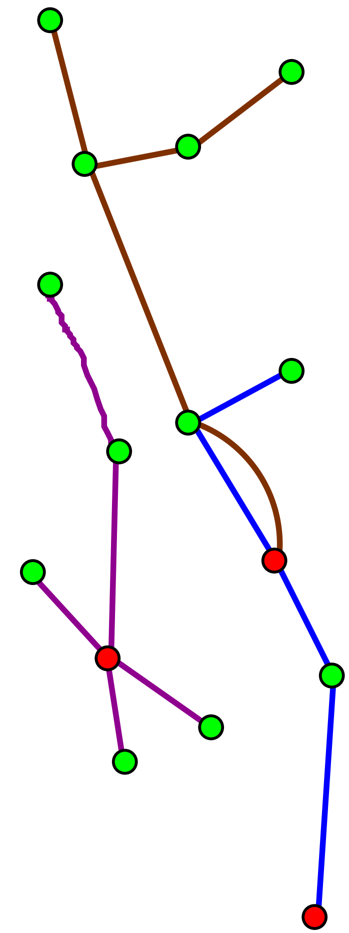
motivation : agents start their tour in different locations.



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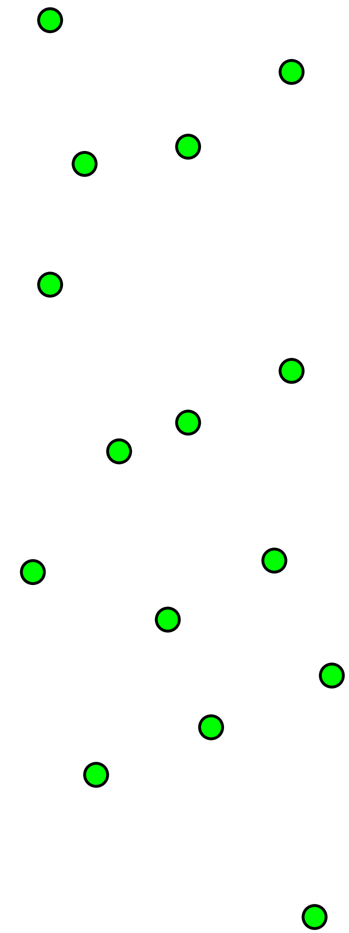

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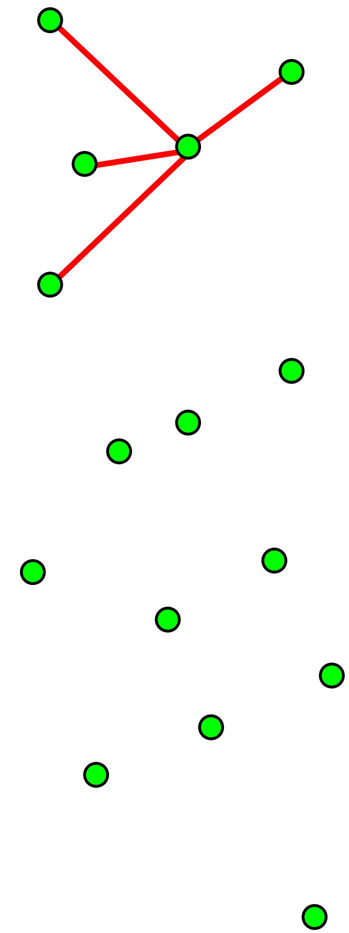

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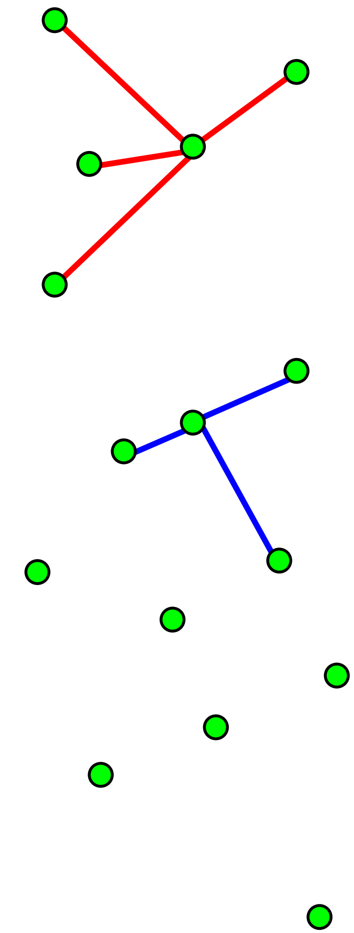

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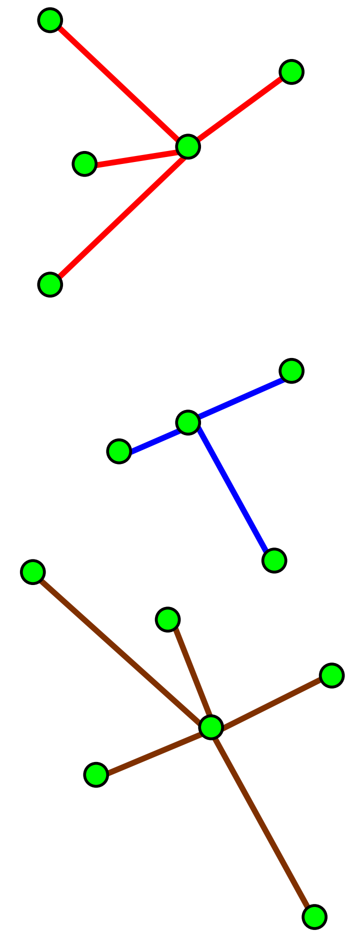

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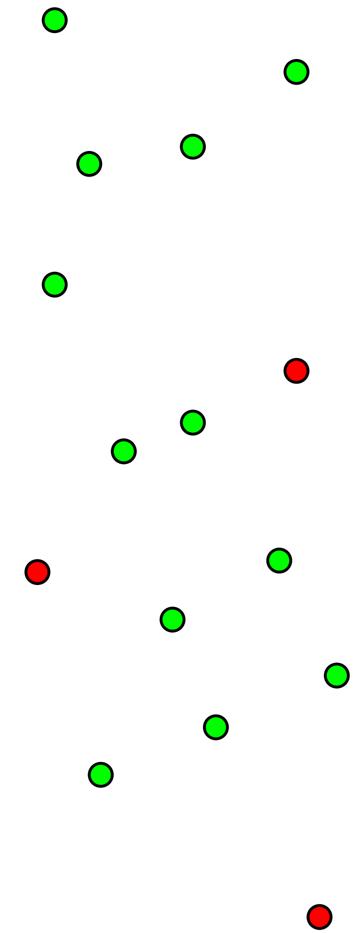

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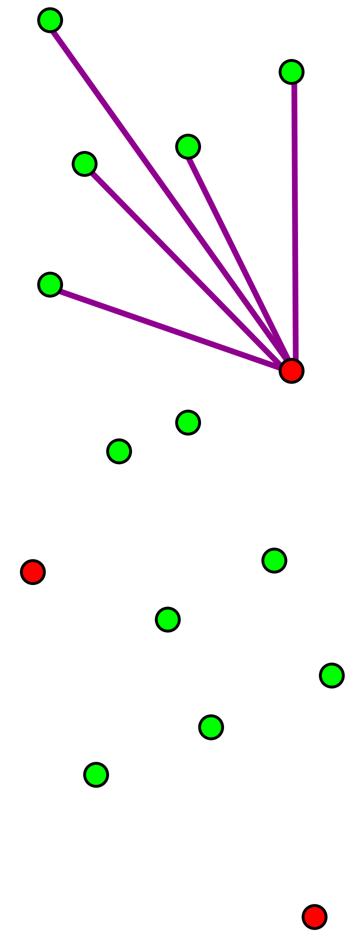

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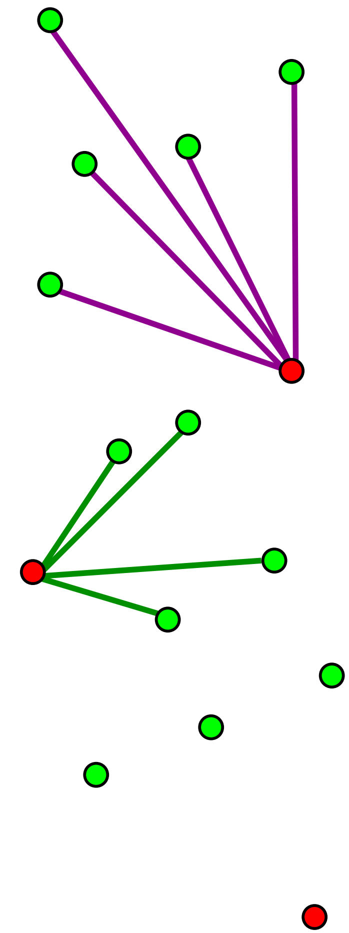

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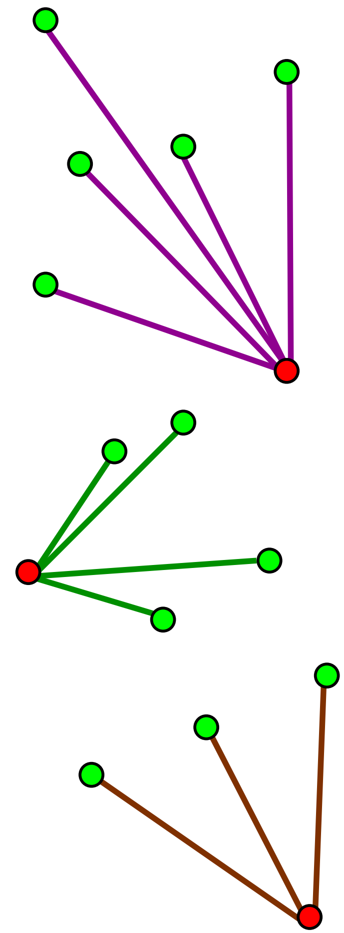

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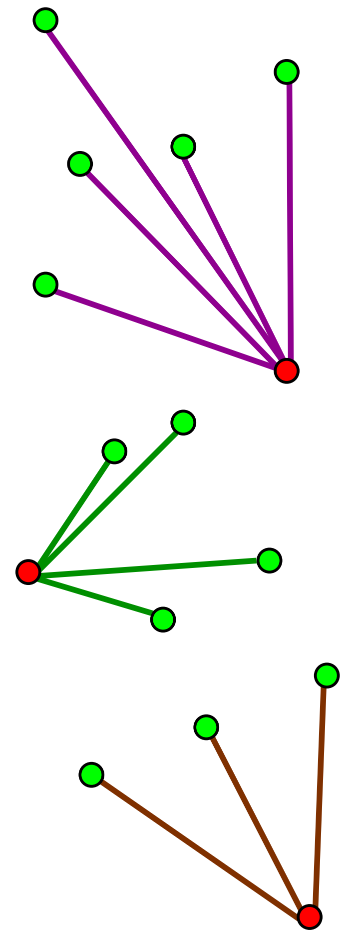

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motivation : agents must return to base after each visit.

Related work

- k-Traveling Repairman: Cover with tours, $O(1)$-approx minimize average latency. [Fakcharoenphol, Harrelson, Rao 2003]

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- Vehicle Routing: Vast amount of work, e.g. Survey [Toth, Vigo, 2002]

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  - Rooted version: equivalent to min. makespan of k machines and n jobs. 2-approximation of [Shmoys & Tardos, 1993].

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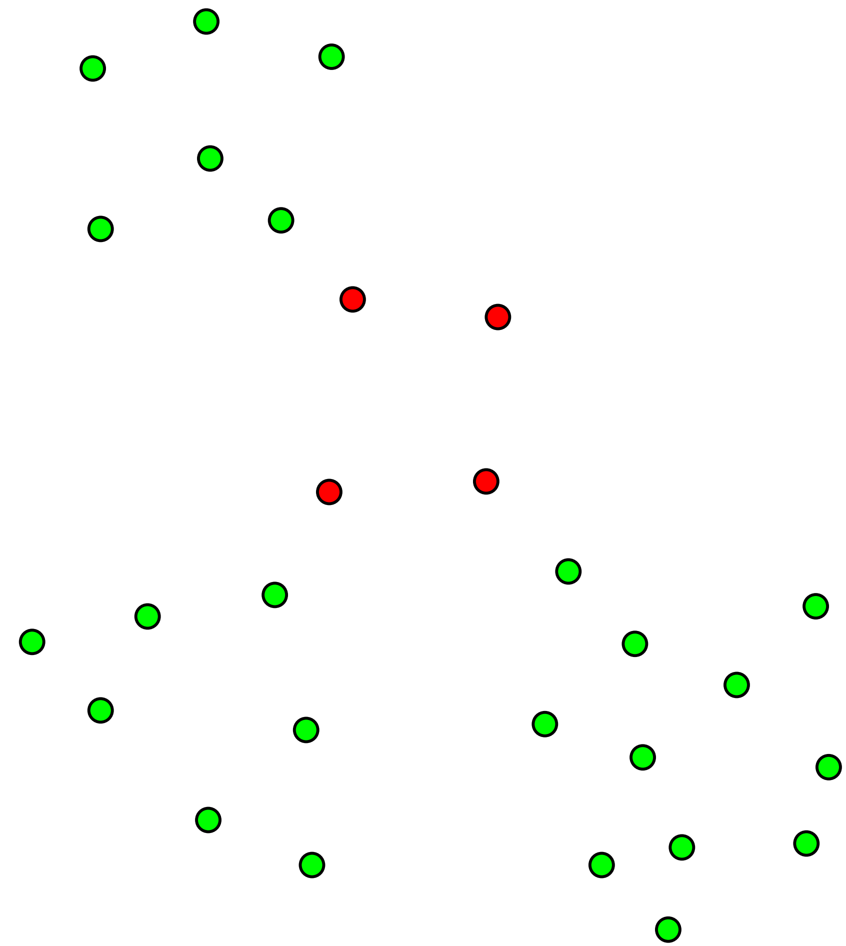

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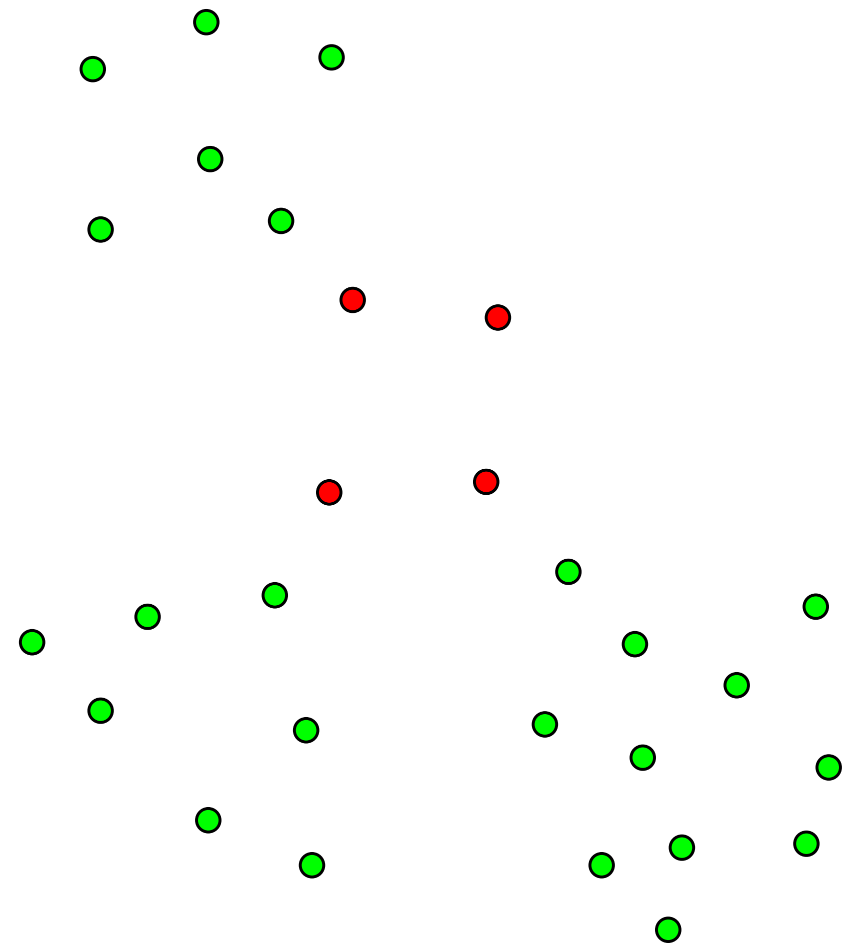

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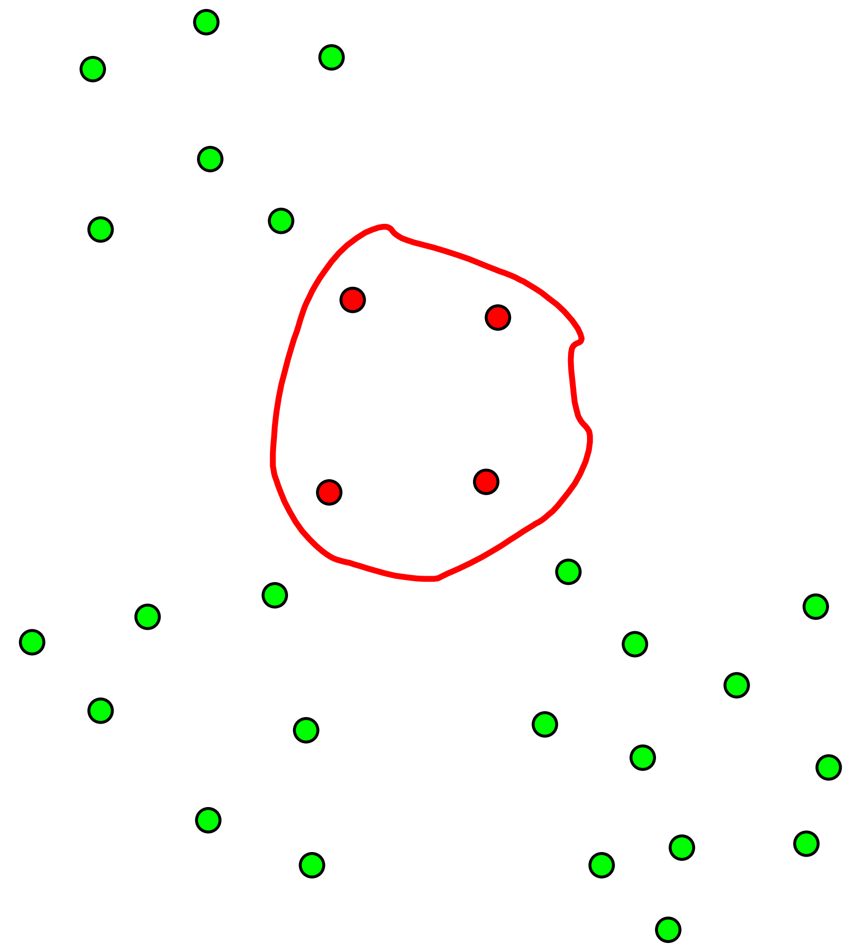

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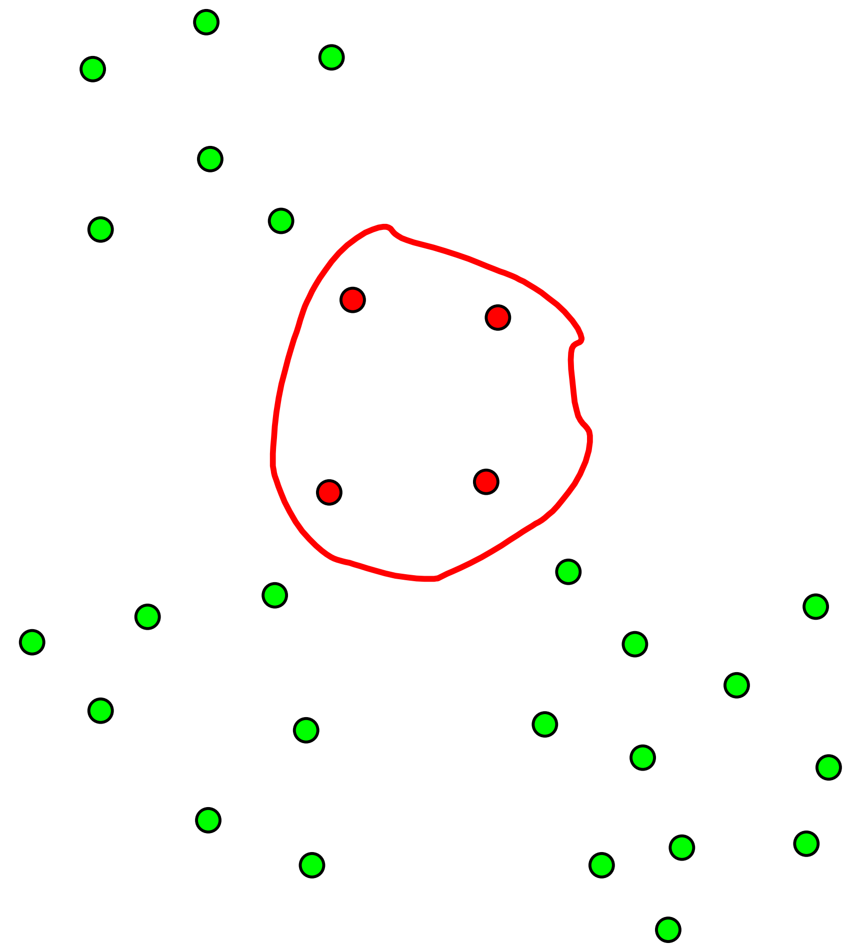

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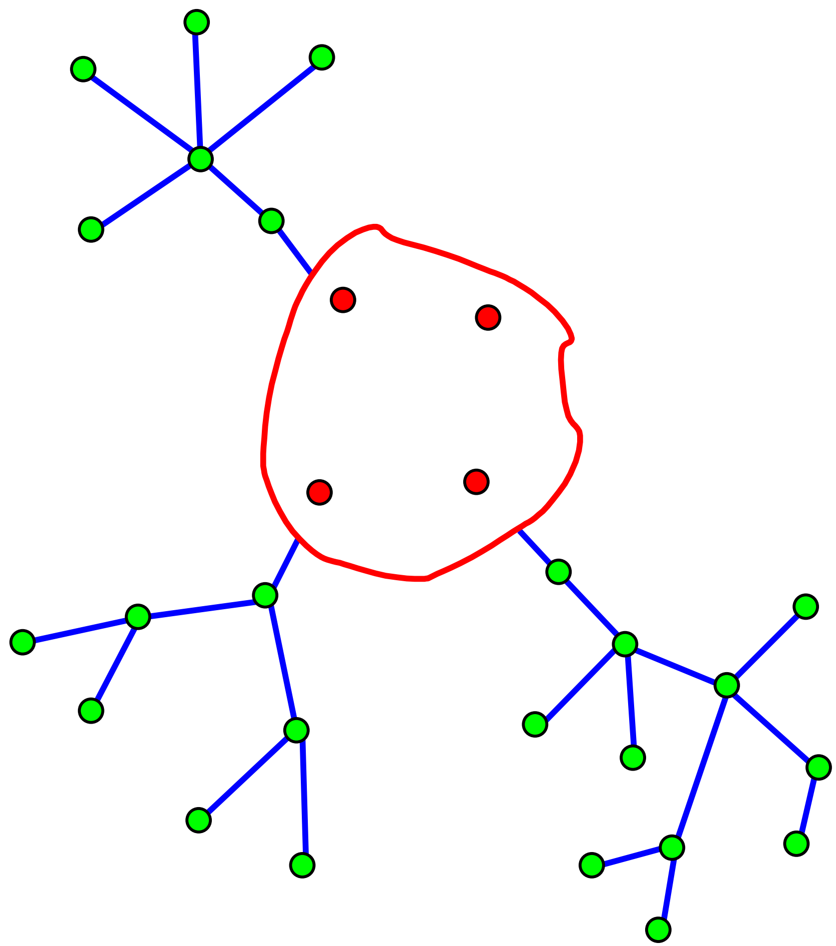

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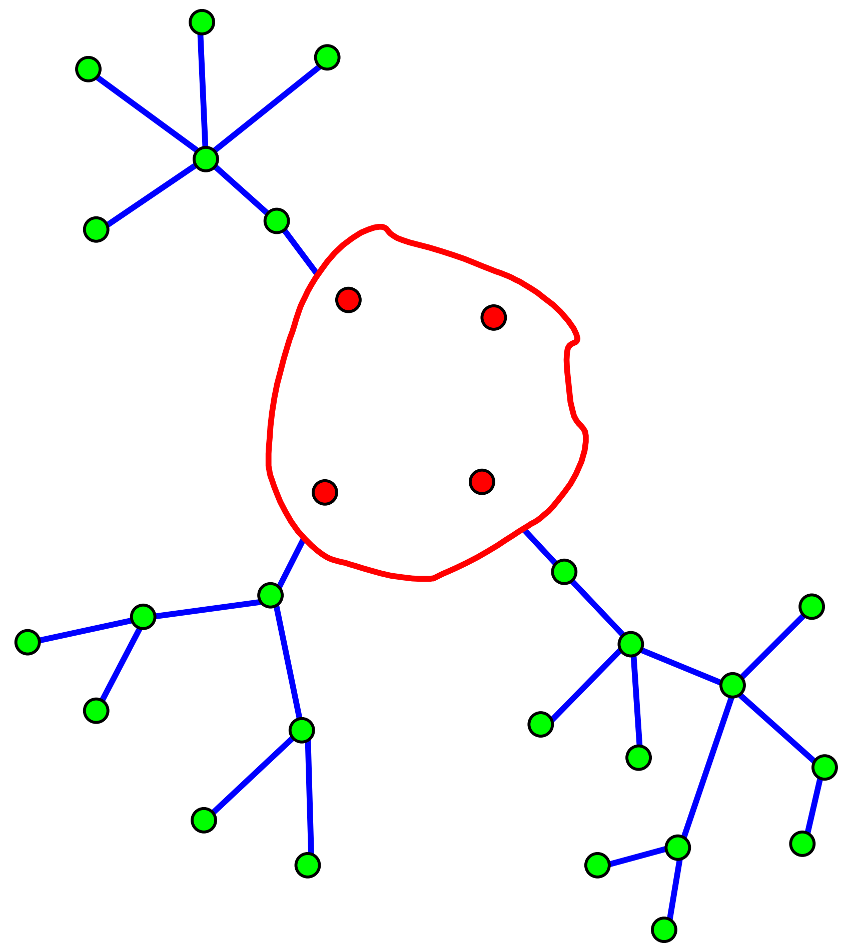

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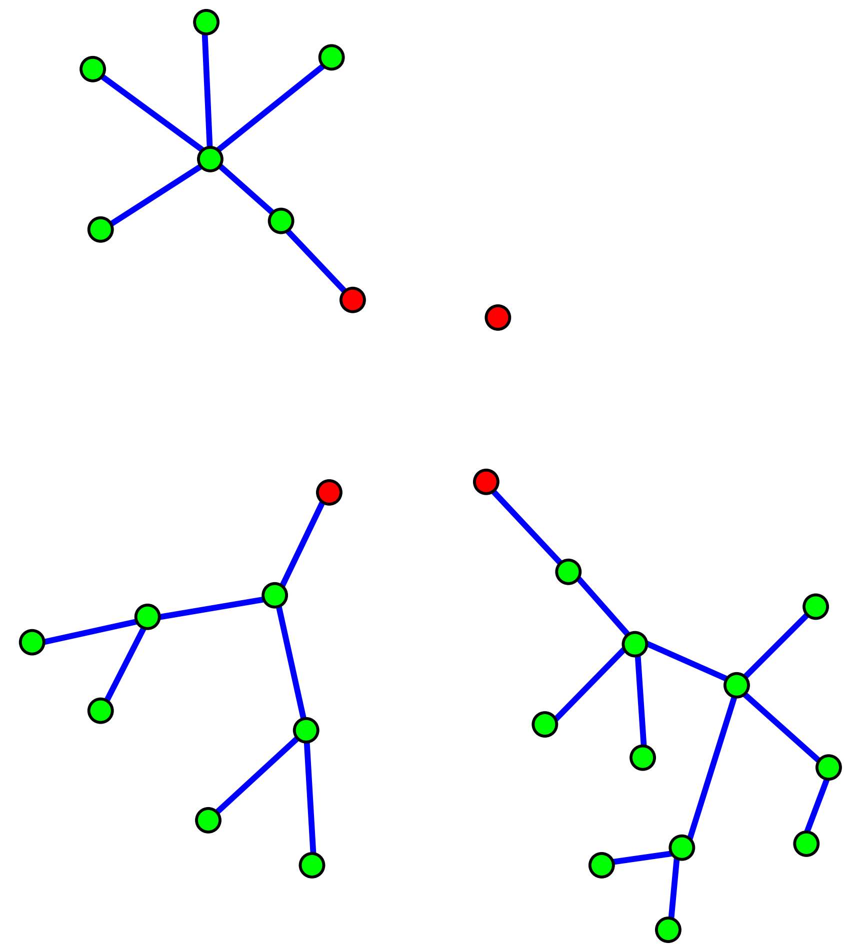

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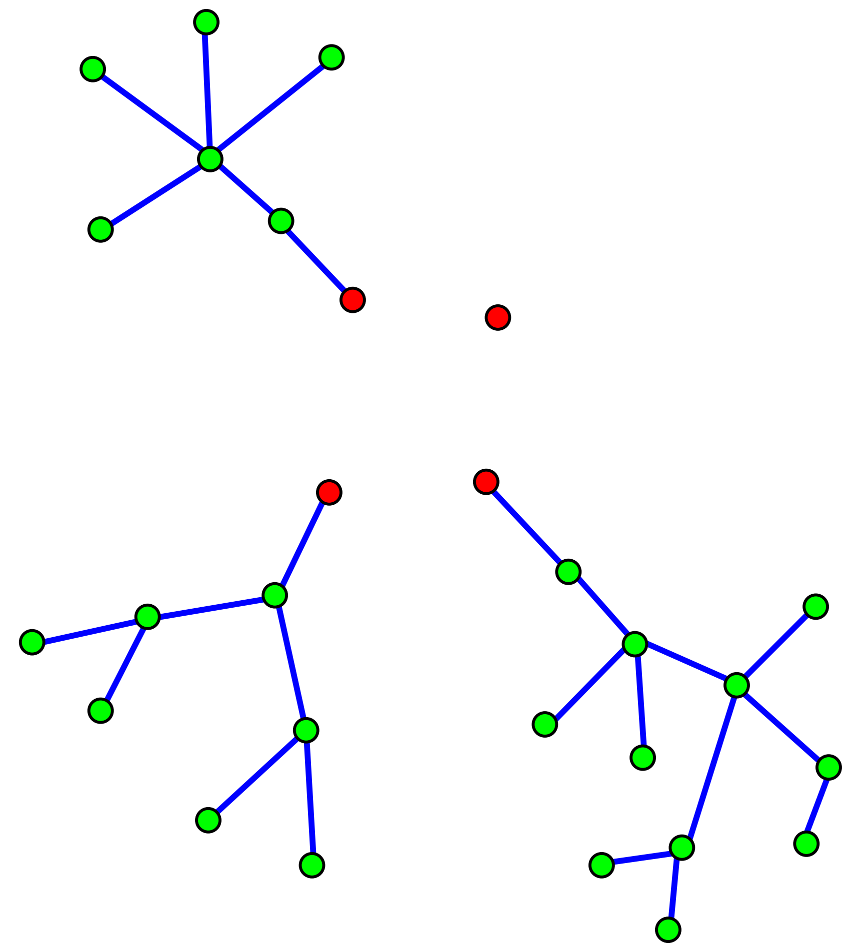

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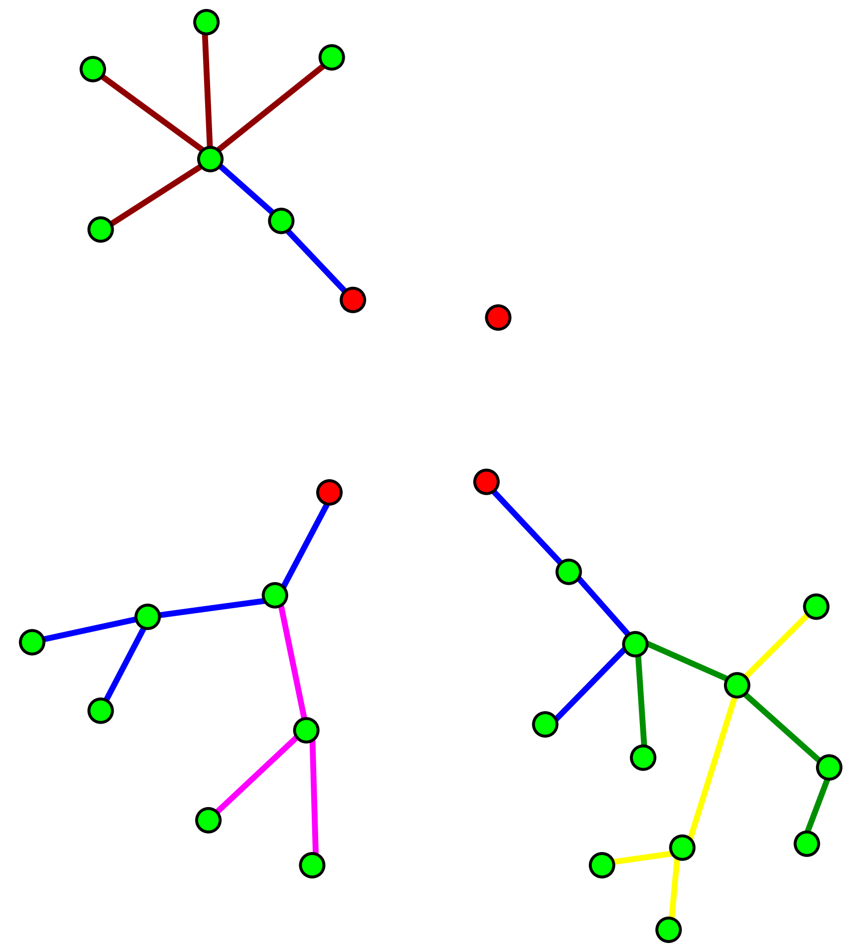

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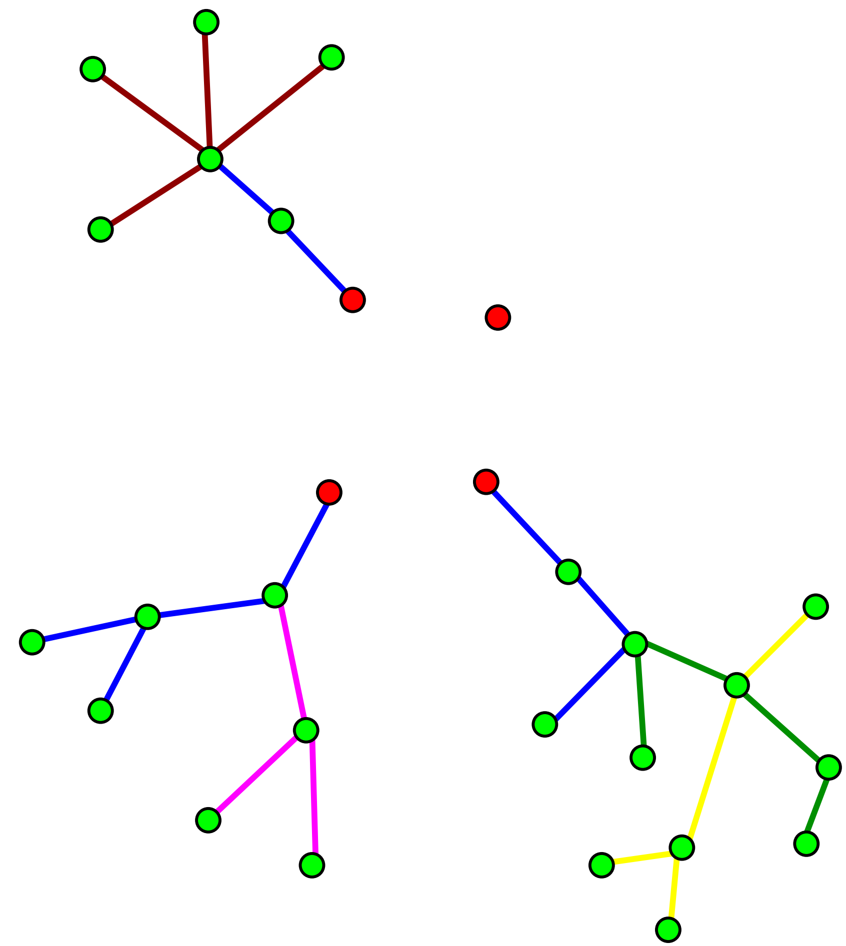

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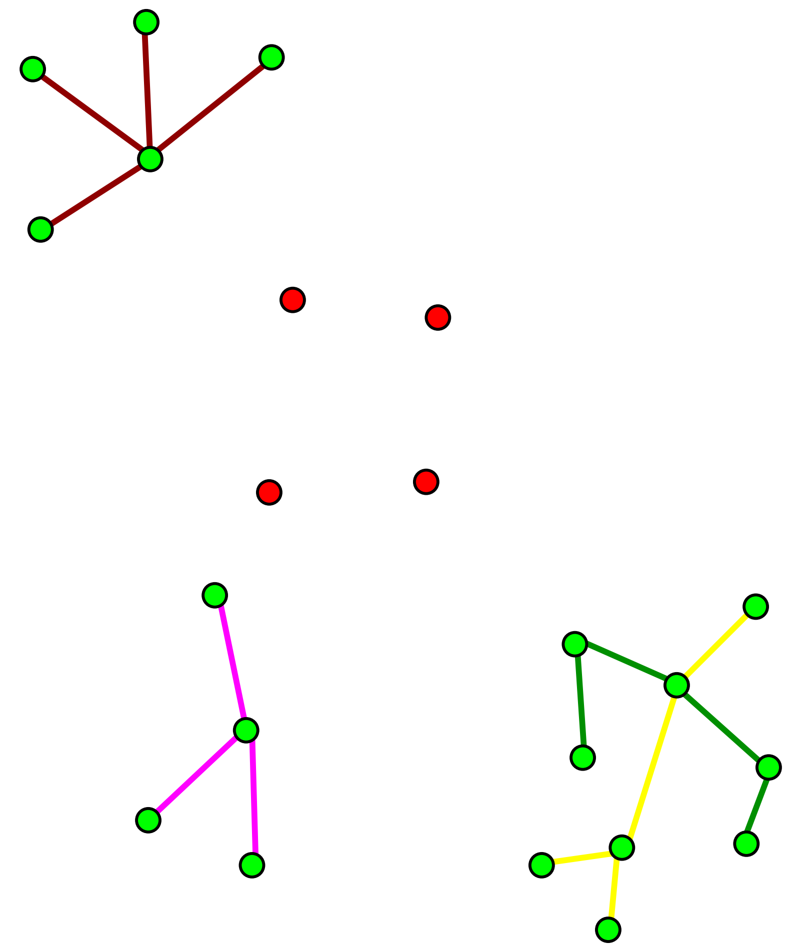

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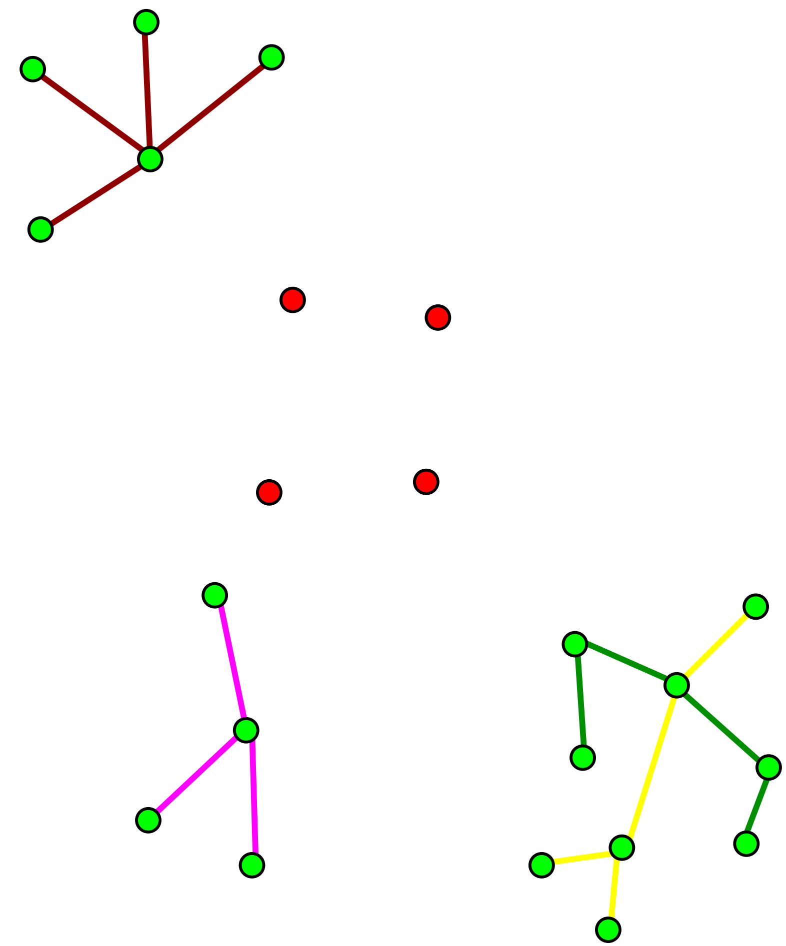

approximation algorithm: k-rooted tree cover

Input: graph, roots, and B - "guess" of opt. cost.

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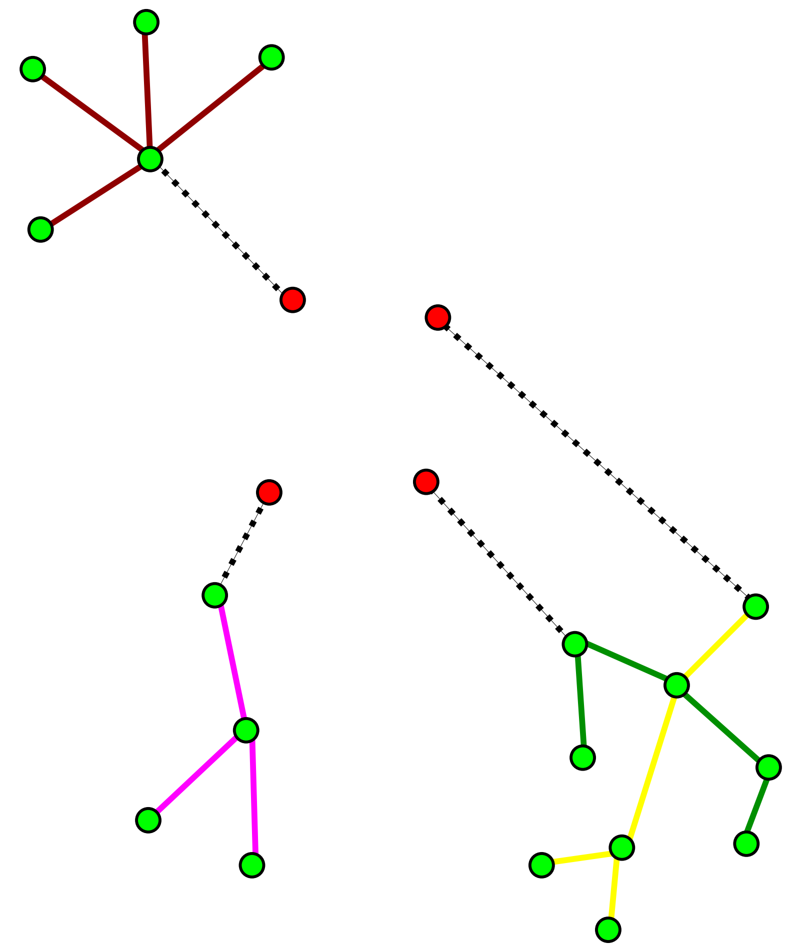

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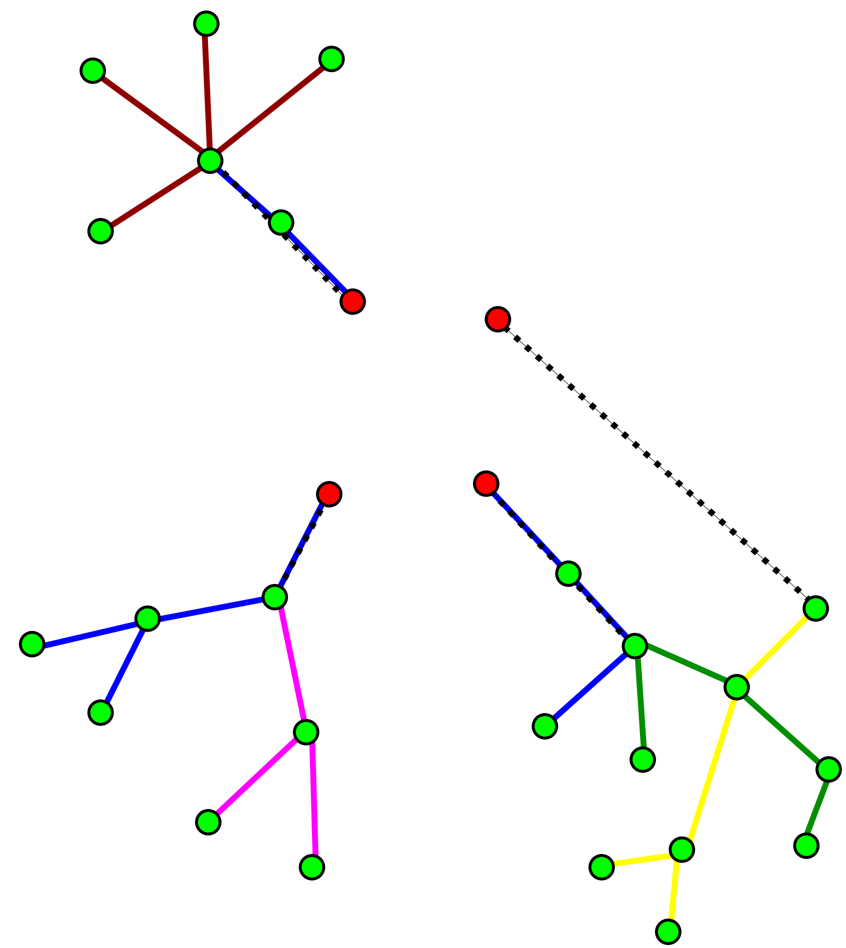

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$\Rightarrow$ weight of every tree in solution is $< 4 \cdot B$. QED

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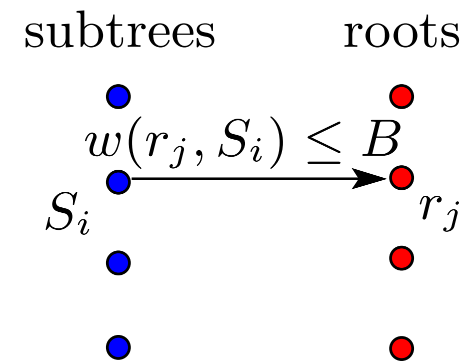

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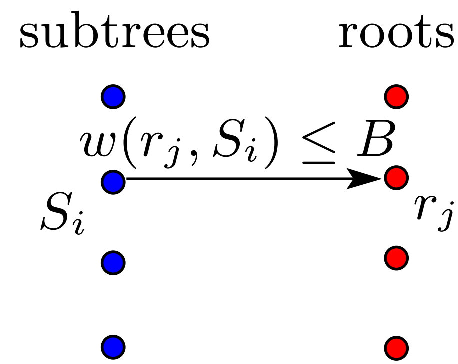

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$\exists \mathcal{S} \subseteq$ subtrees : $|\mathcal{S}| > |N(\mathcal{S})|$.

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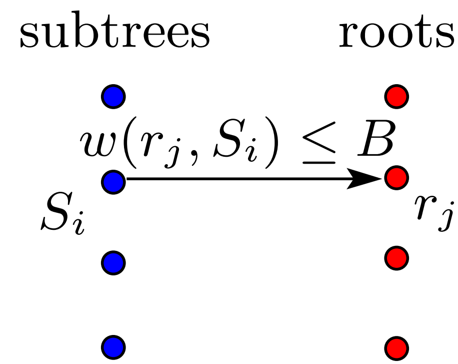

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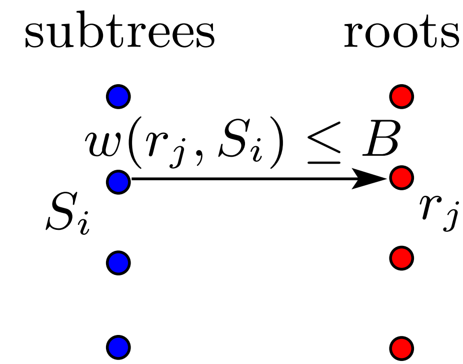

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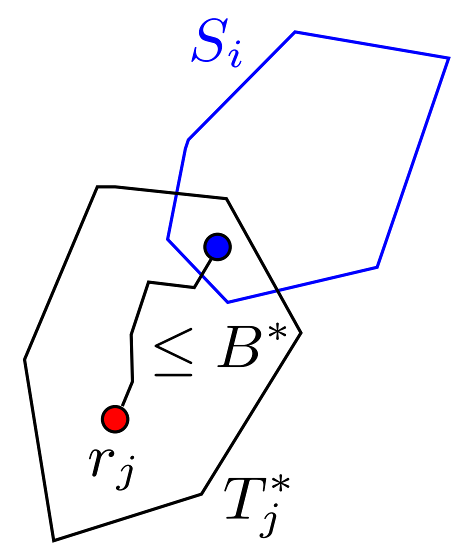

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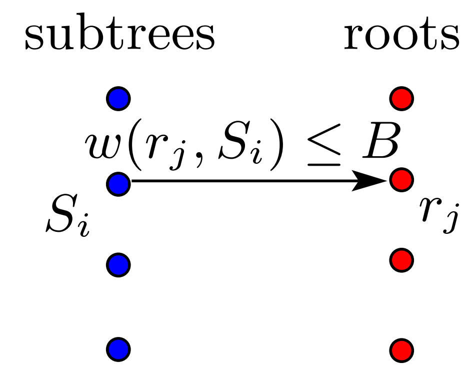

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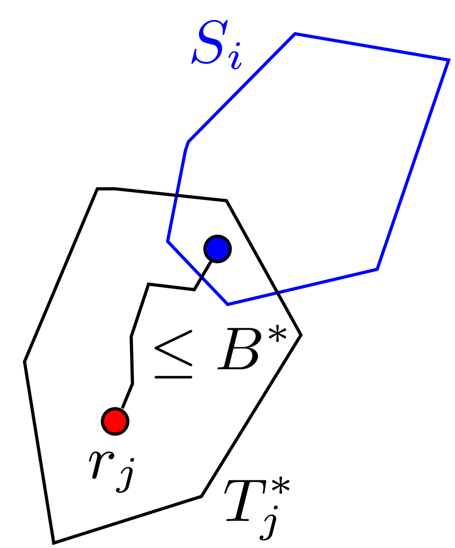

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But

$$T' \triangleq MST + \mathcal{T}^*(\mathcal{S}) - \mathcal{S}$$

is a spanning tree and $w(T') < w(MST)$, contradiction. QED

Algorithm for Unrooted k-tree cover

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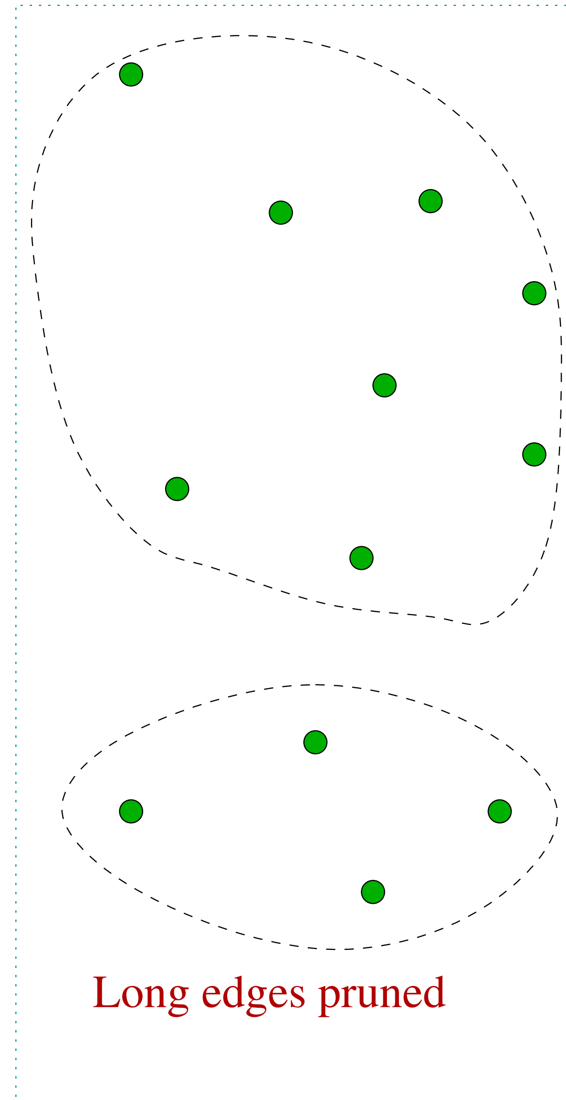

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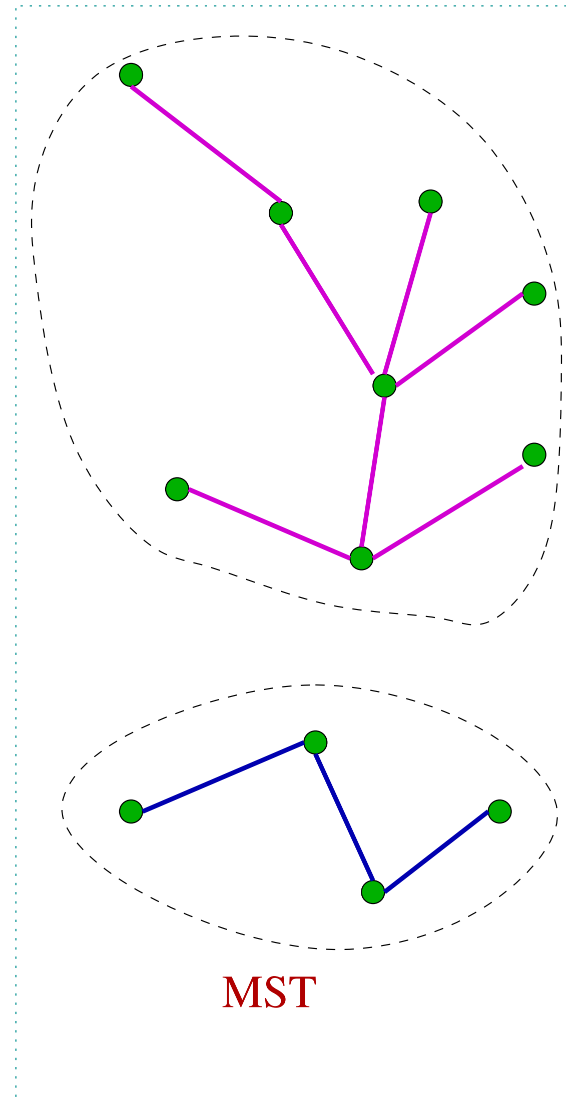

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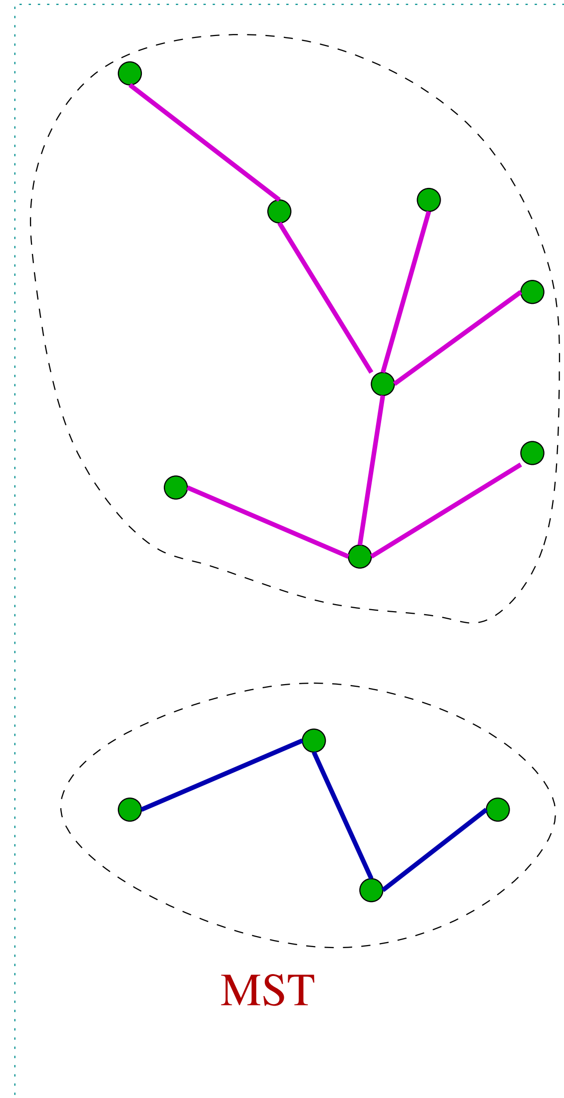

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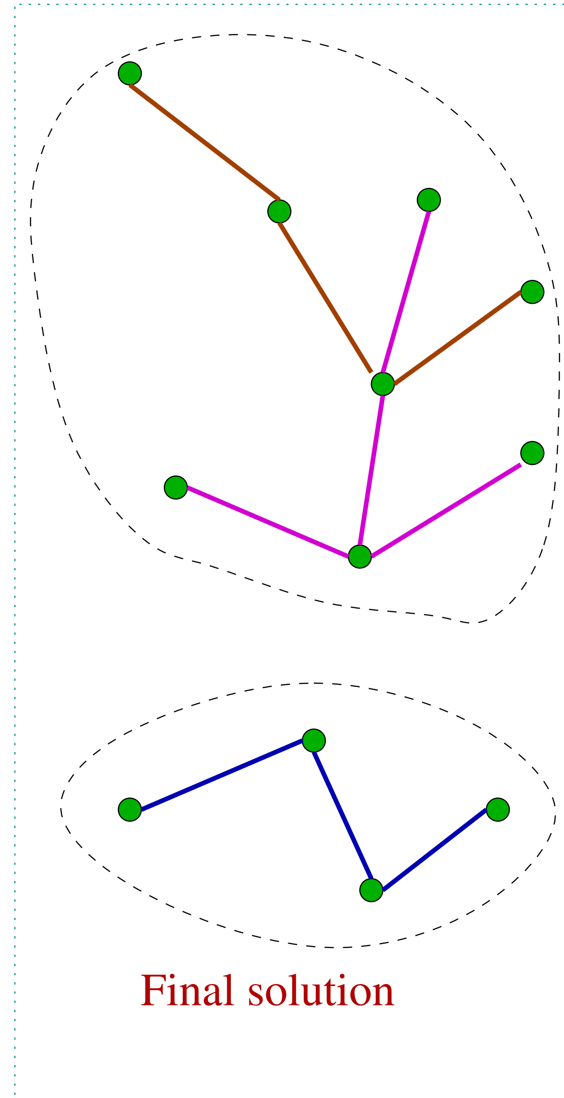

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Alternatively, if $B^* \leq B$, then $k_i + 1 \leq k_i^*$ for all i.

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$\square$