Covering Graphs Using Trees and Stars

Guy Even (Tel-Aviv), Naveen Garg (Delhi),

and

Jochen Könemann, R. Ravi and A. Sinha (Pittsburgh)

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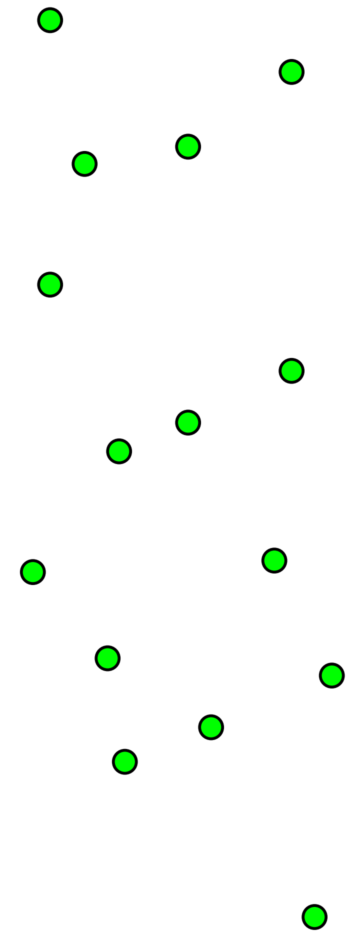

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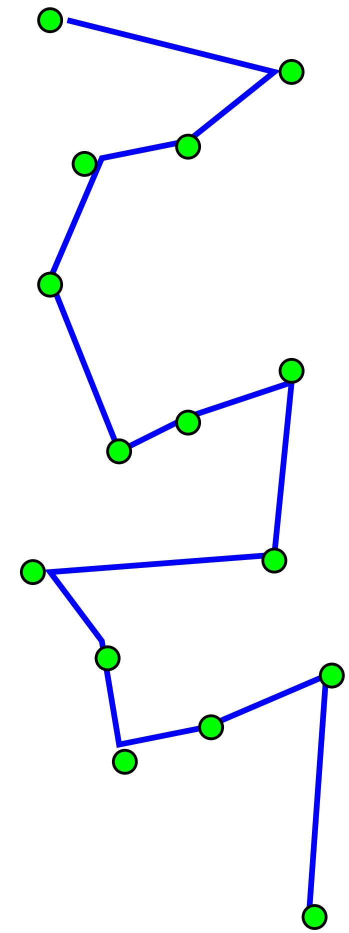

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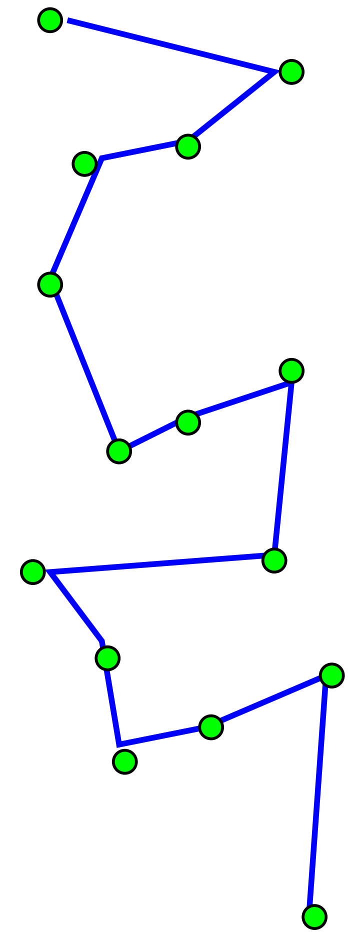

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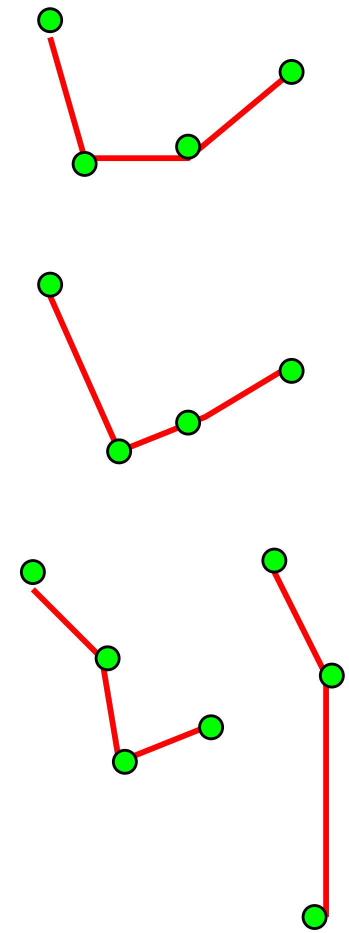

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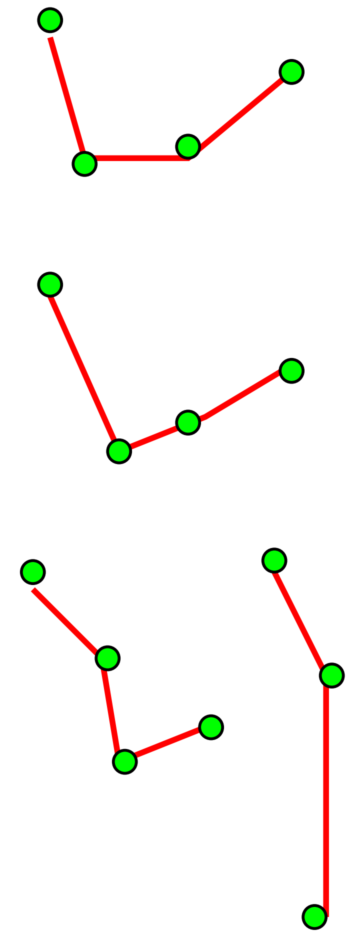

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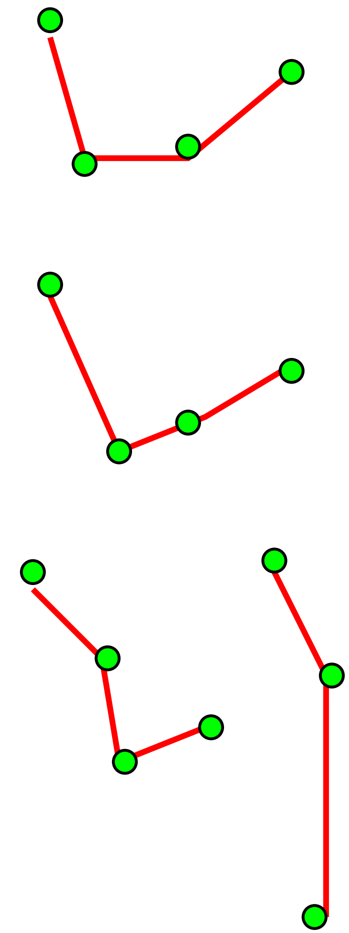

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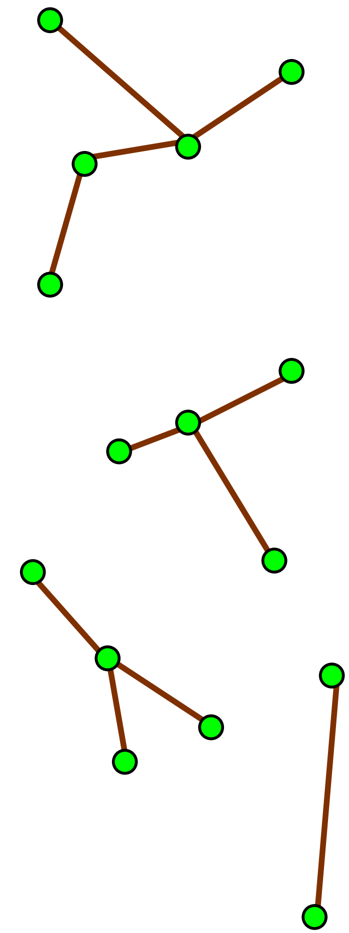
- MST is a constant ratio approx of a min tour $\Rightarrow$ k-Tree Cover Problem.



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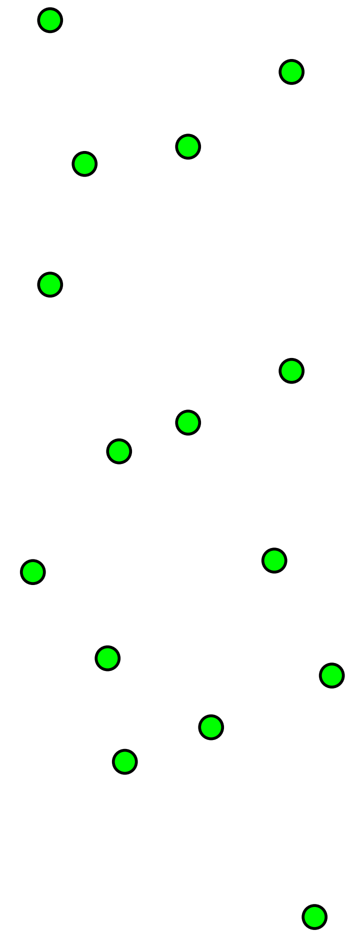

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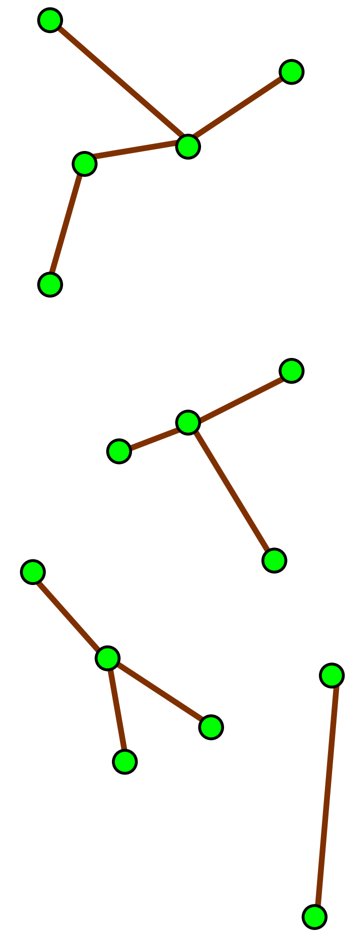

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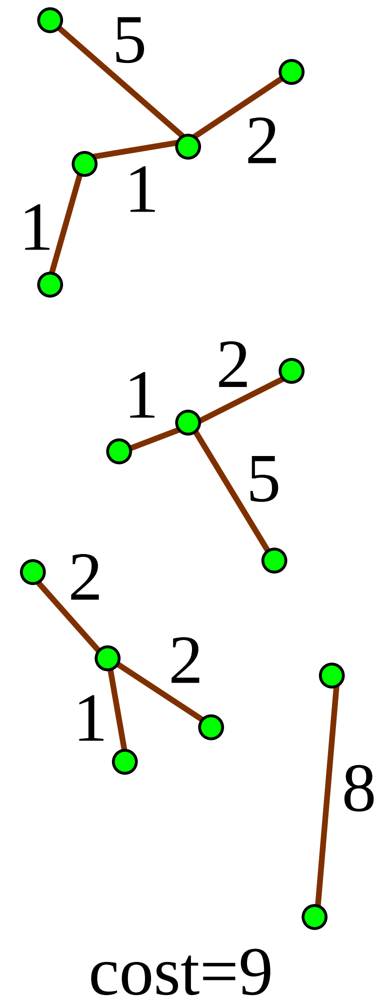

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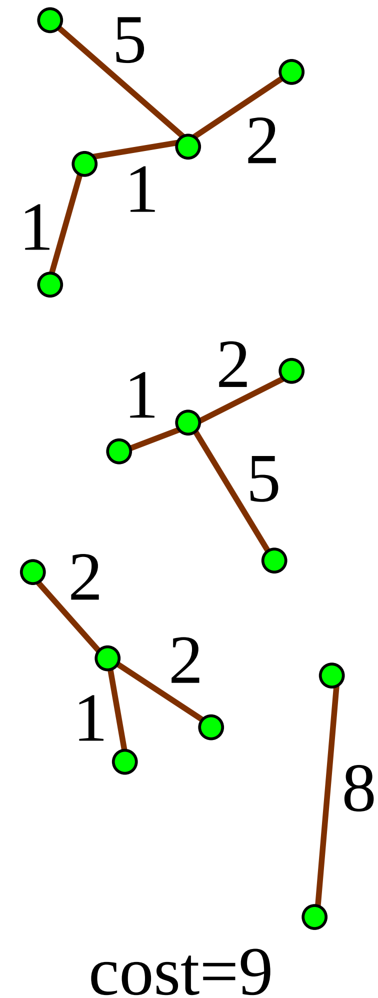

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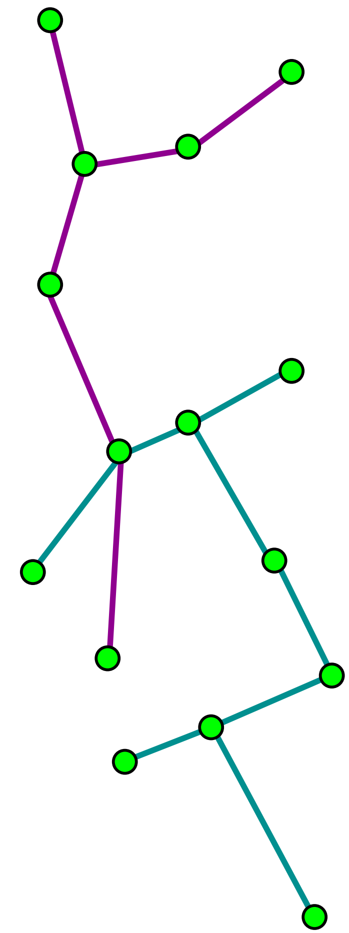
remark : trees may share nodes & edges in a tree cover.



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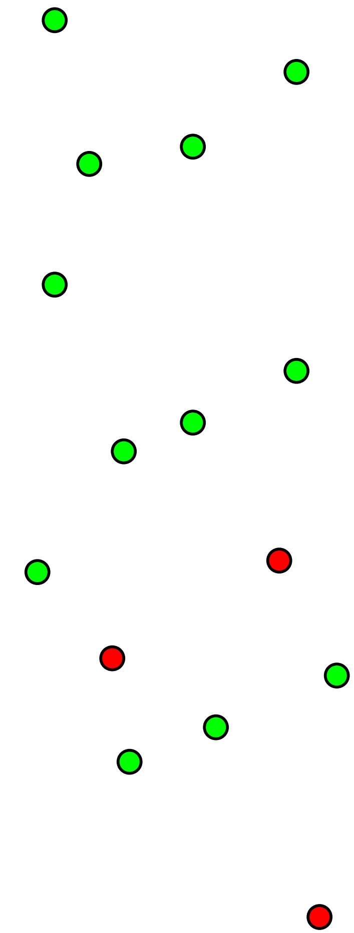

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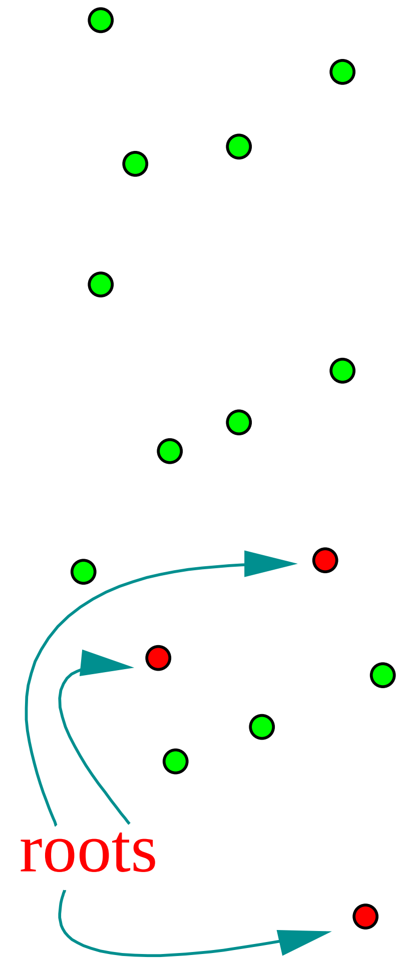

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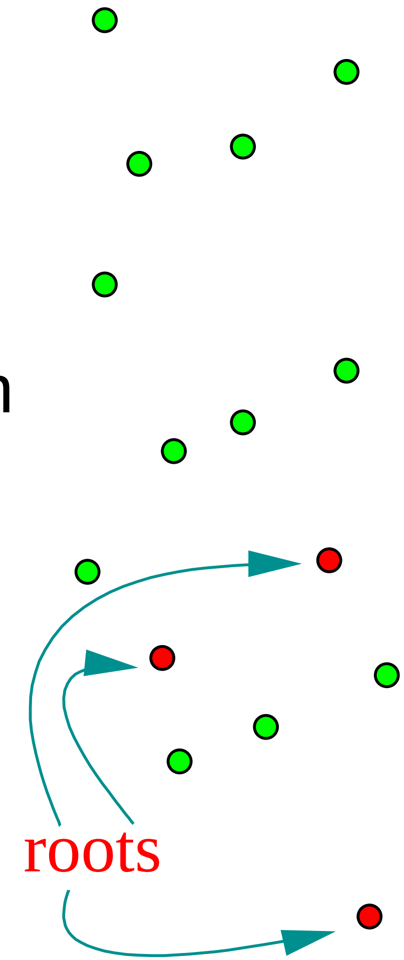

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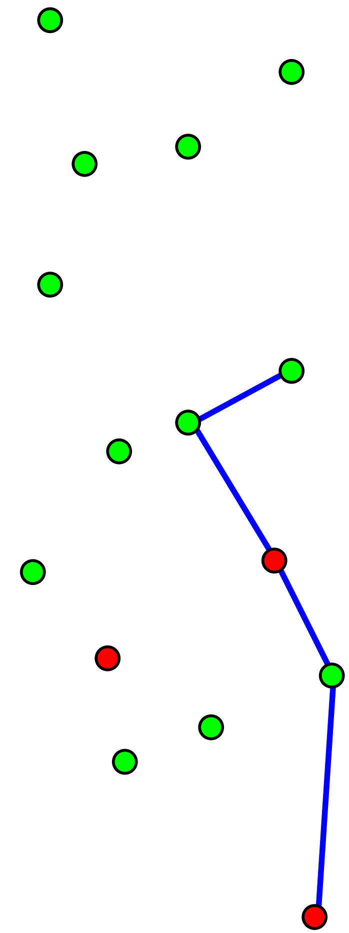

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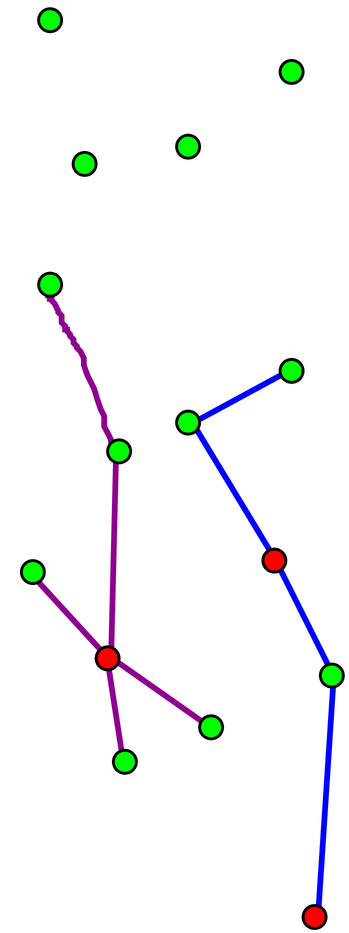

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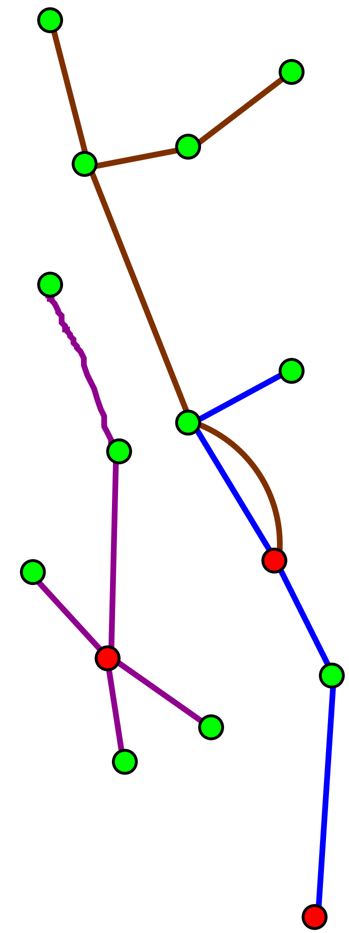

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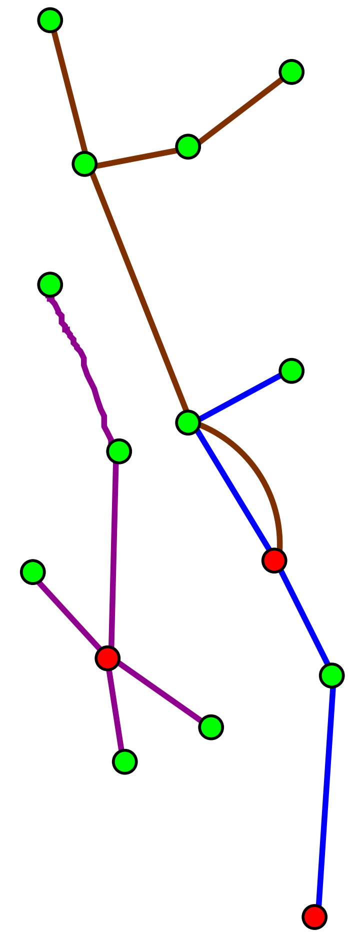
motivation : agents start their tour in different locations.



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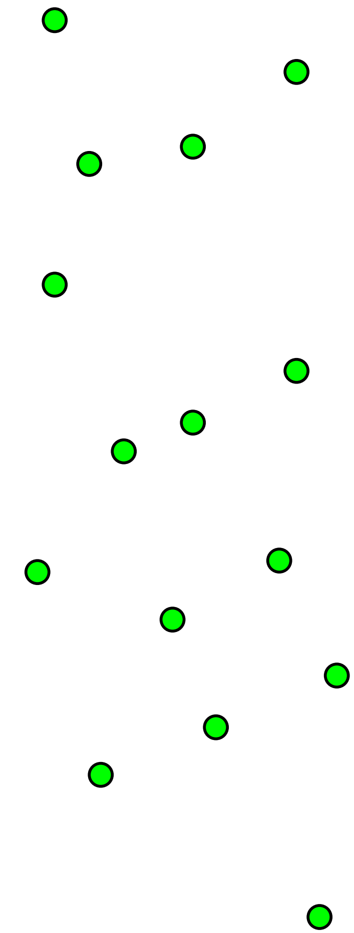

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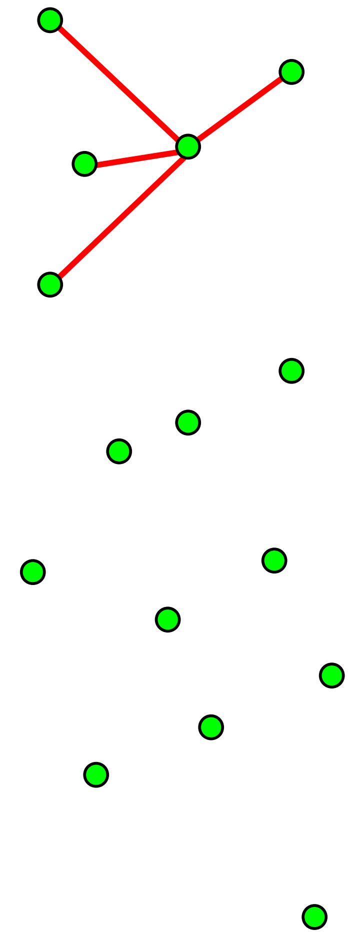

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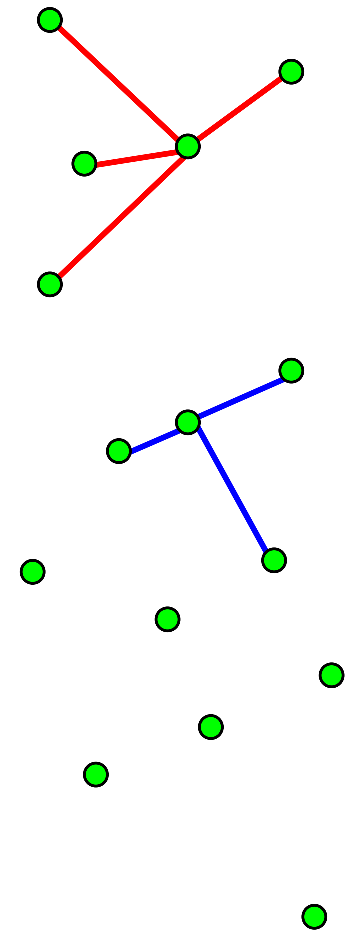

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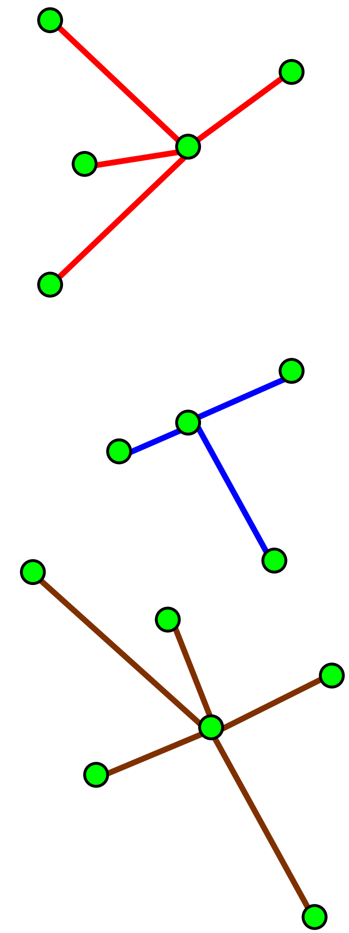

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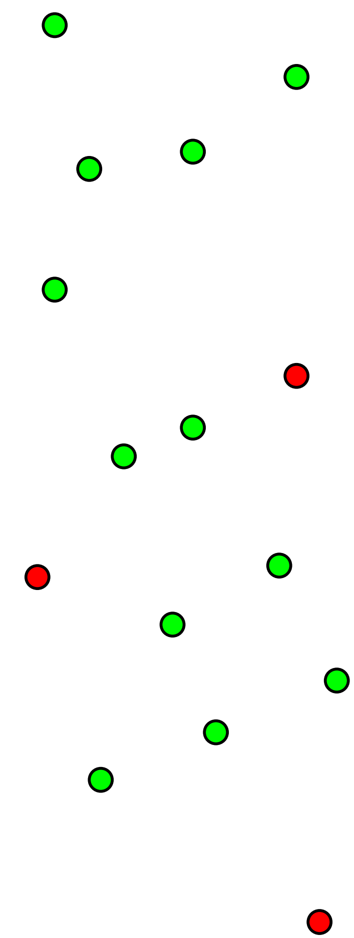

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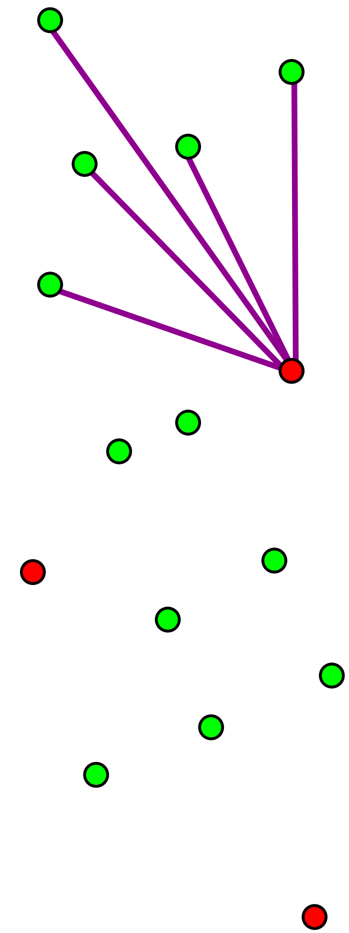

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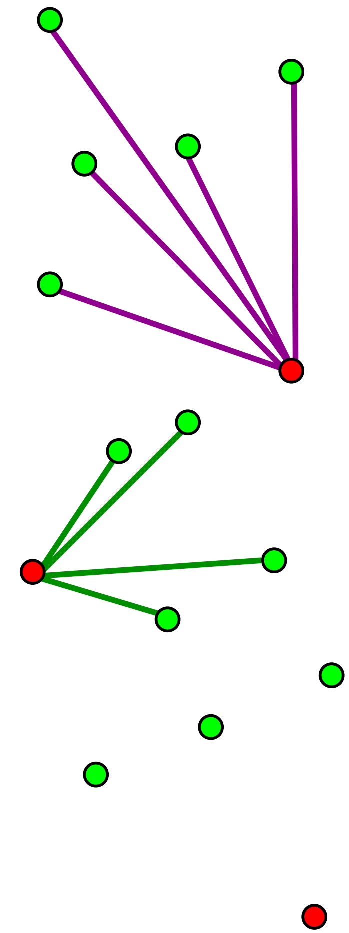

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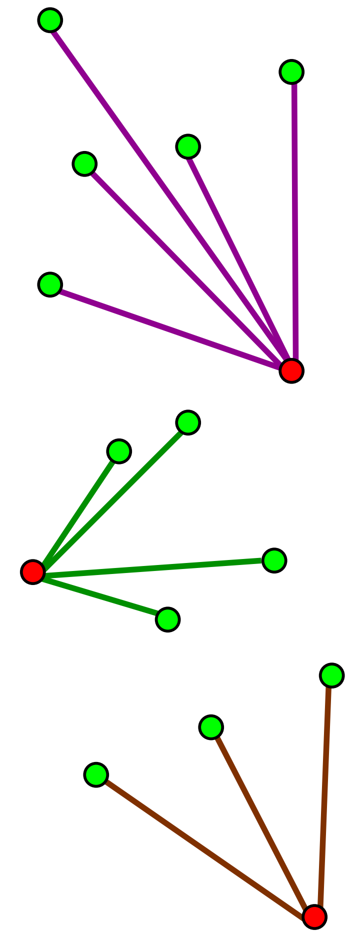

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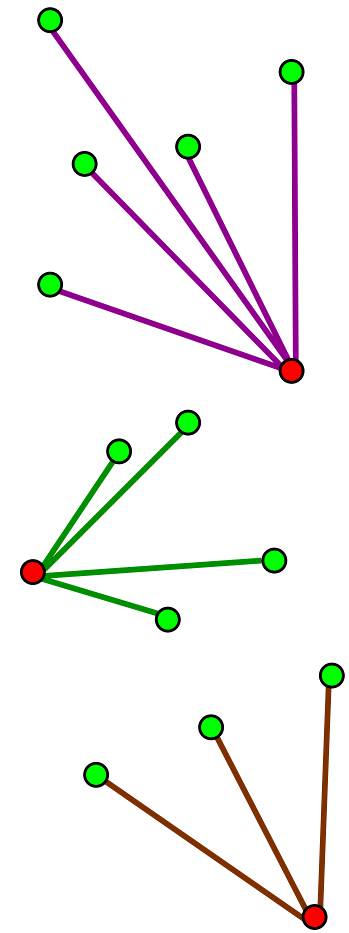
motivation : agents must return to base after each visit.



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- Chandra Chekuri & Amit Kumar - similar results for star covers.

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- k-Traveling Repairman: Cover with tours, $O(1)$-approx minimize average latency. [Fakcharoenphol, Harrelson, Rao 2003]

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- Vehicle Routing: Vast amount of work, e.g. Survey [Toth, Vigo, 2002]

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  - Rooted version: equivalent to min. makespan of k machines and n jobs. 2-approximation of [Shmoys & Tardos, 1993].

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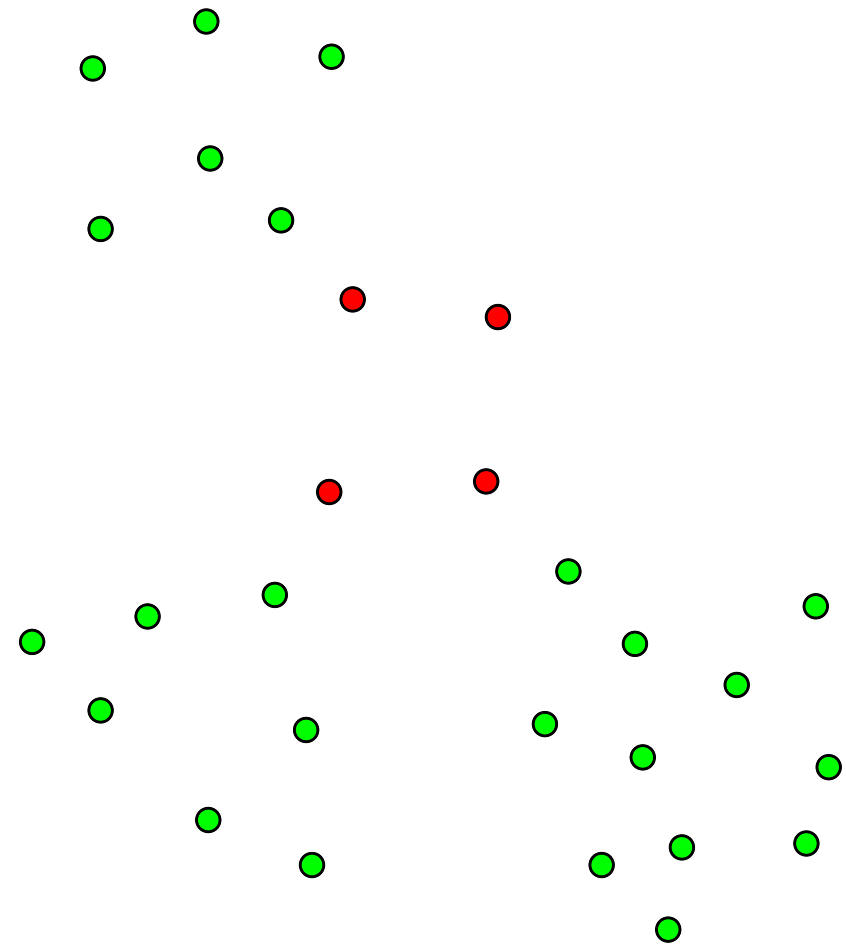
Input: graph, roots, and B - “guess” of opt. cost.



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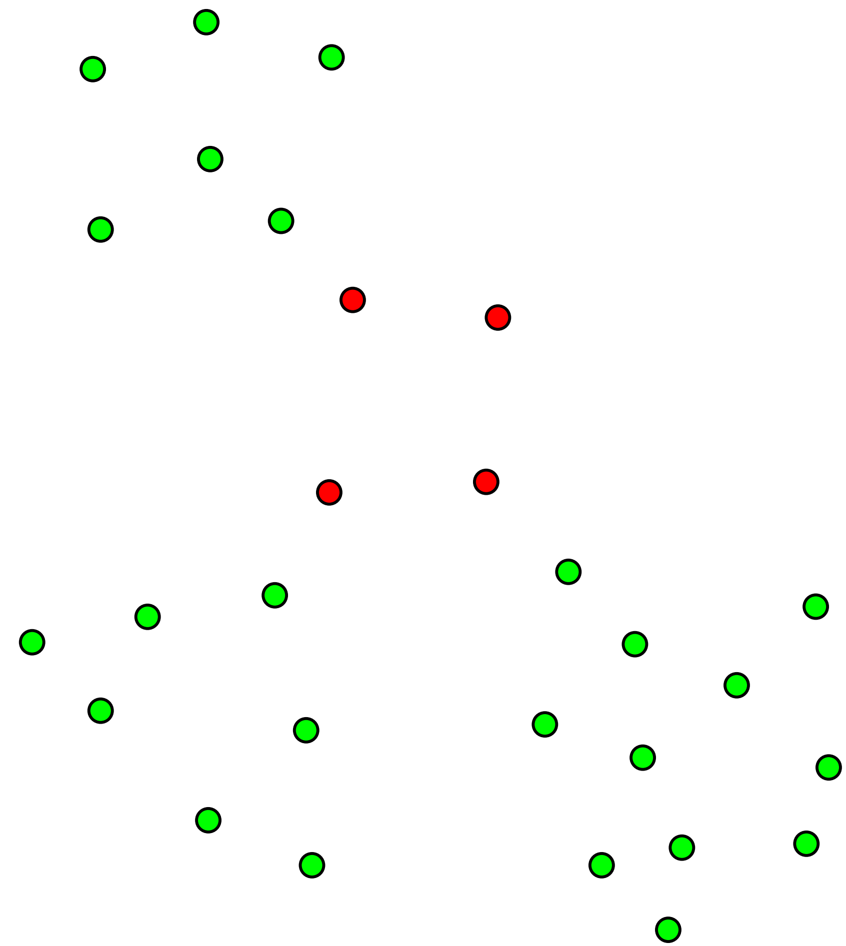

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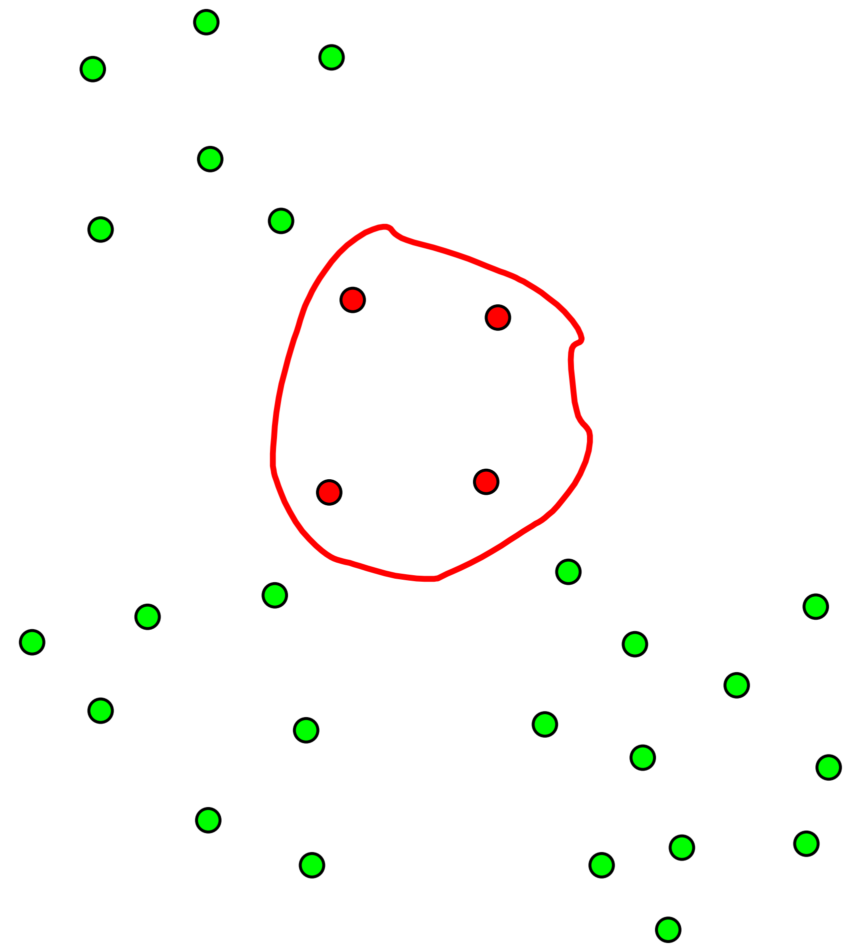

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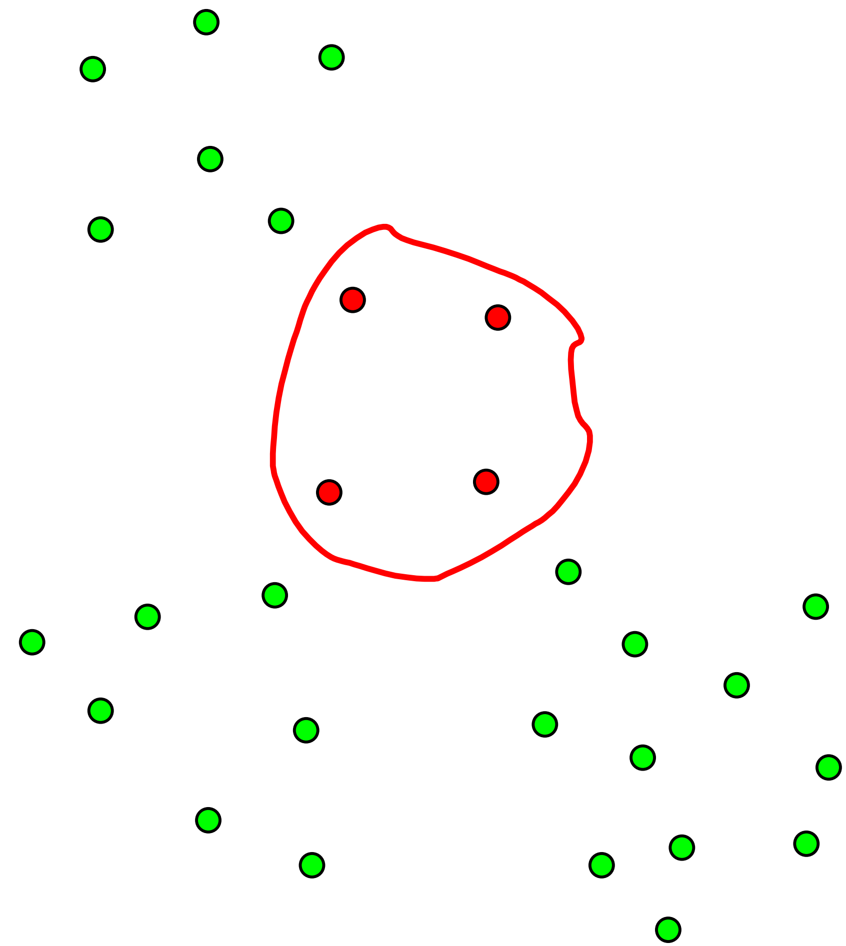

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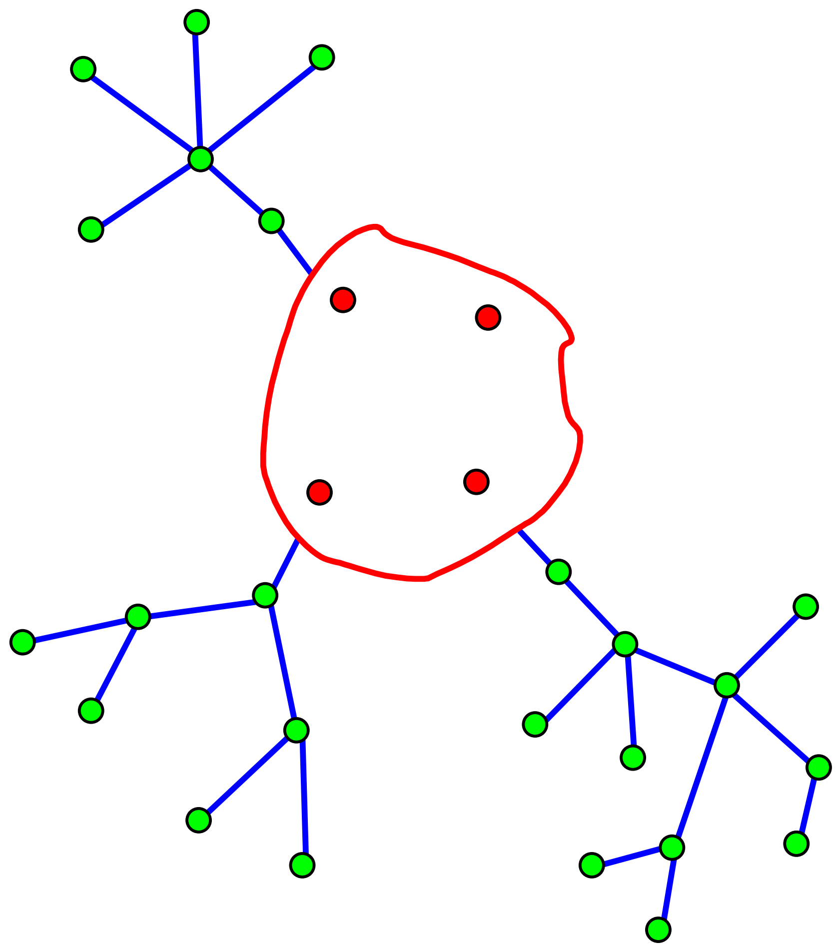

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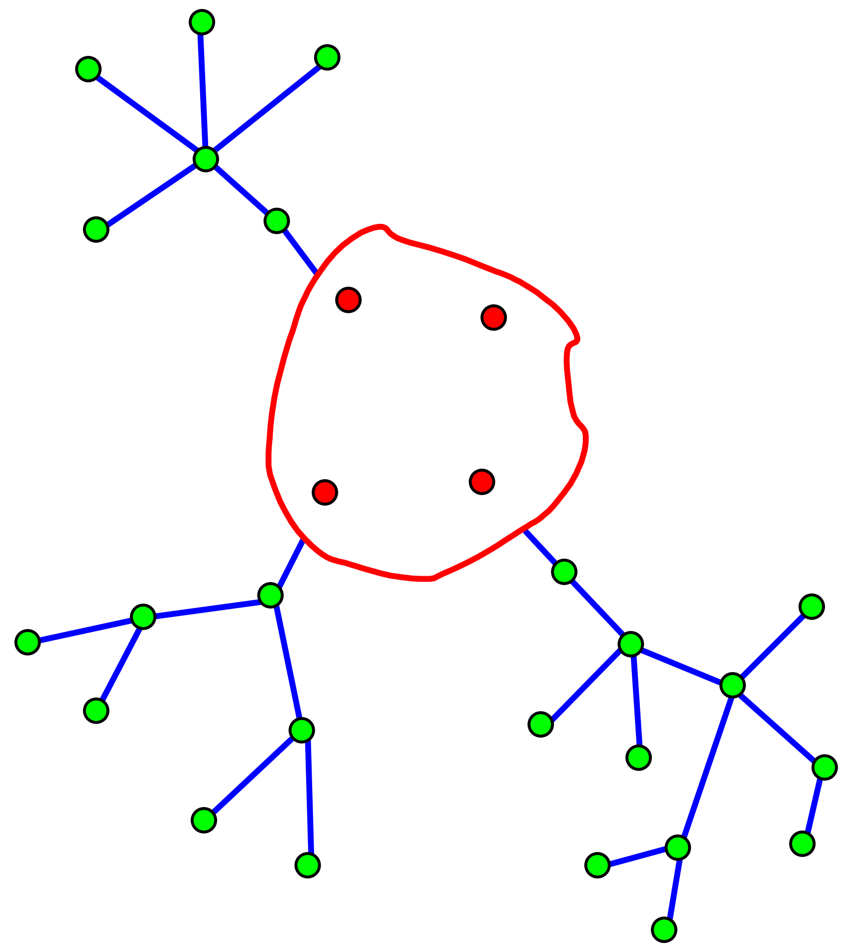

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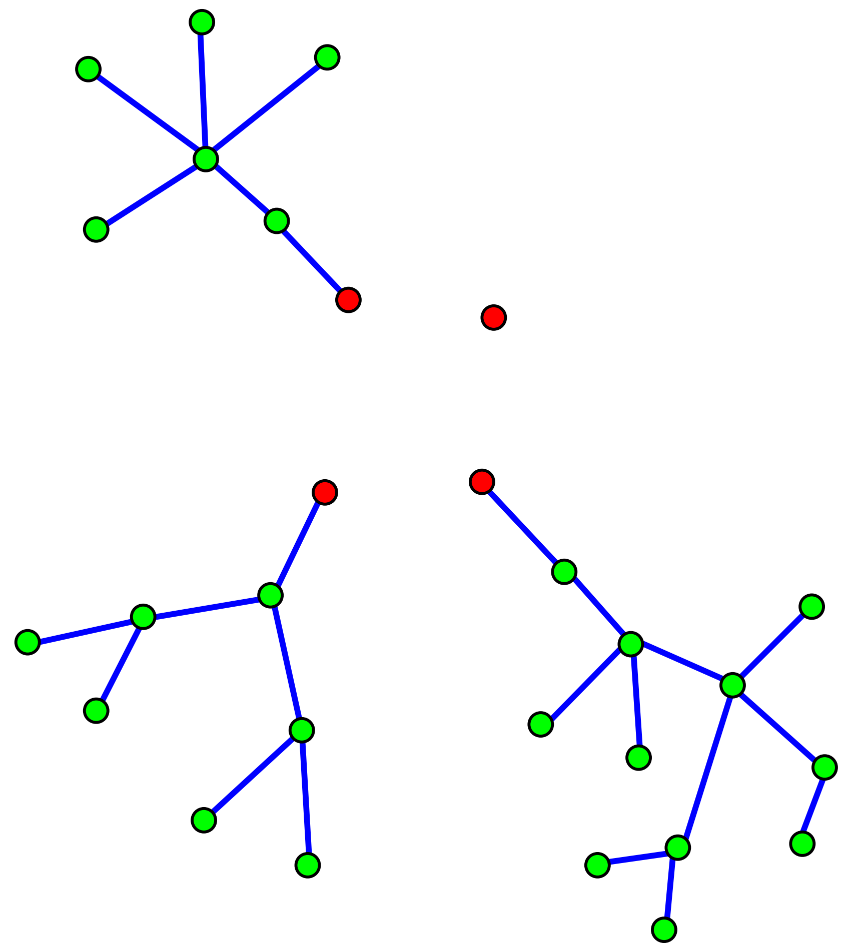

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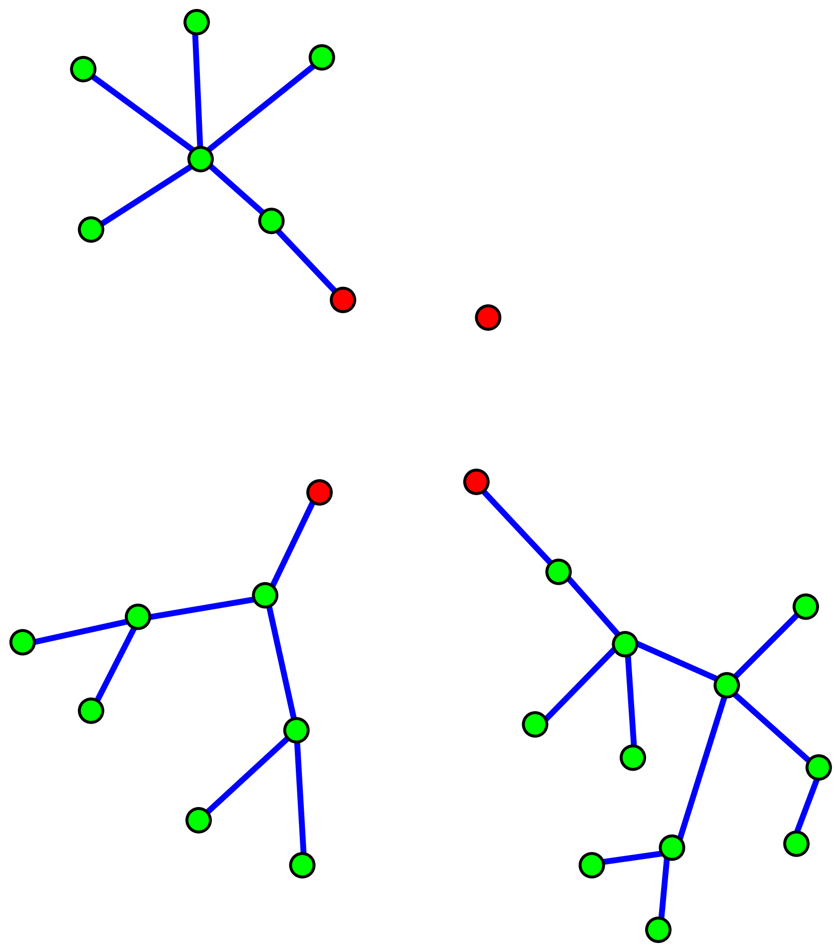

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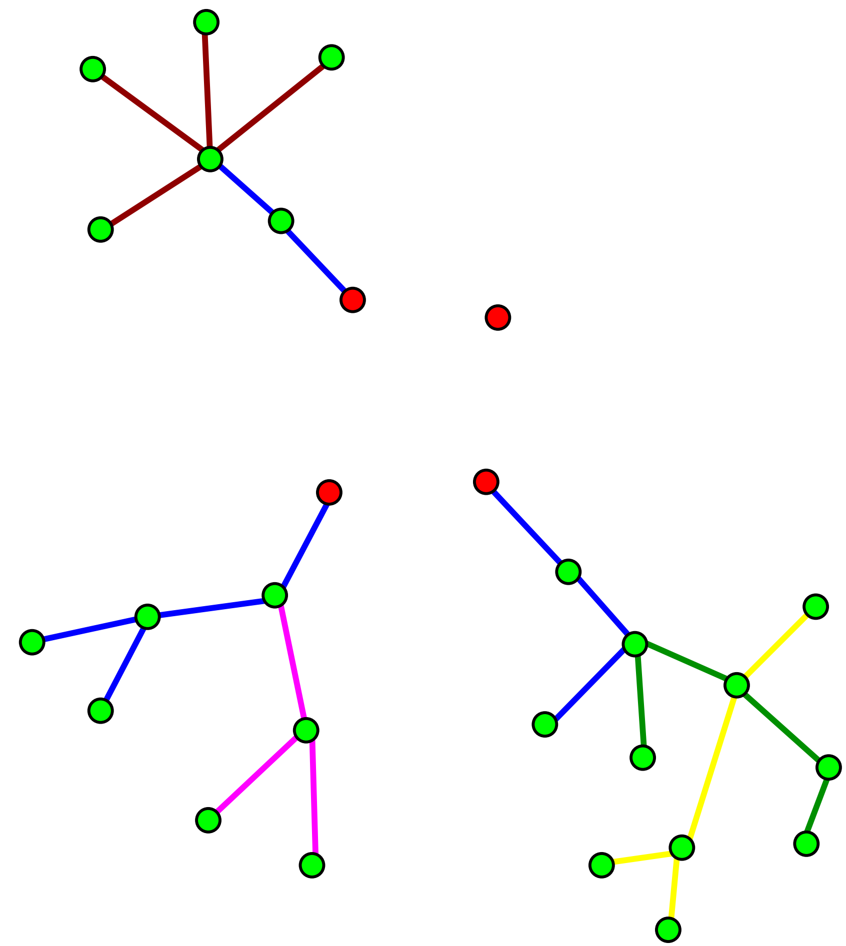

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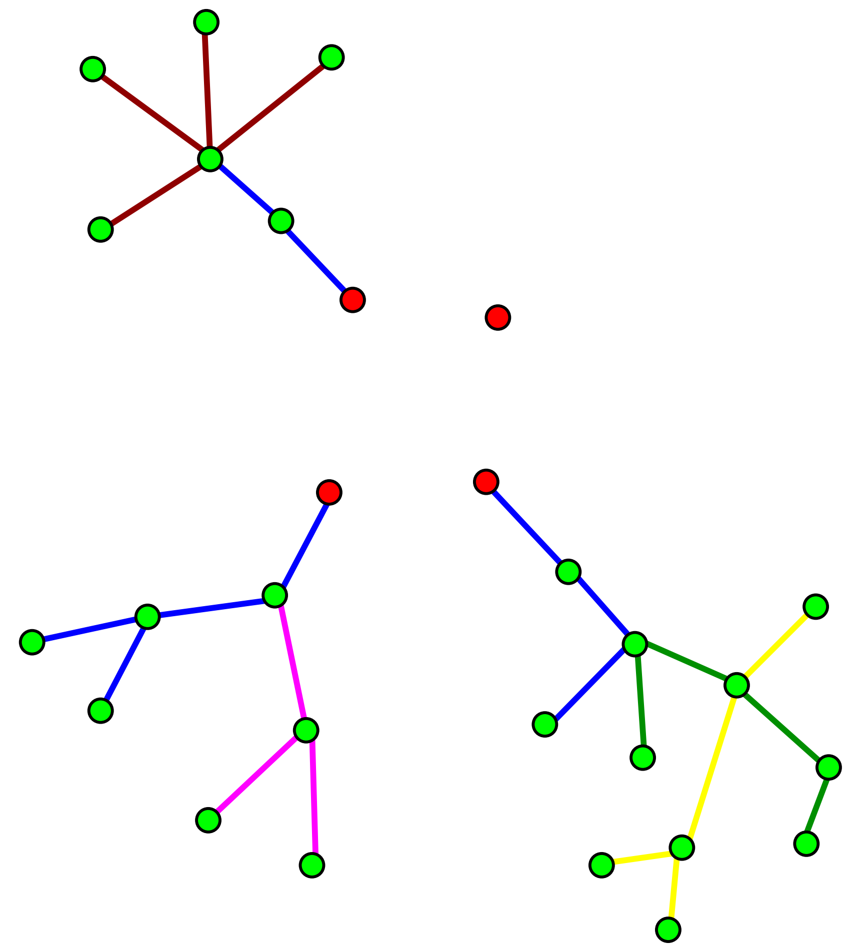

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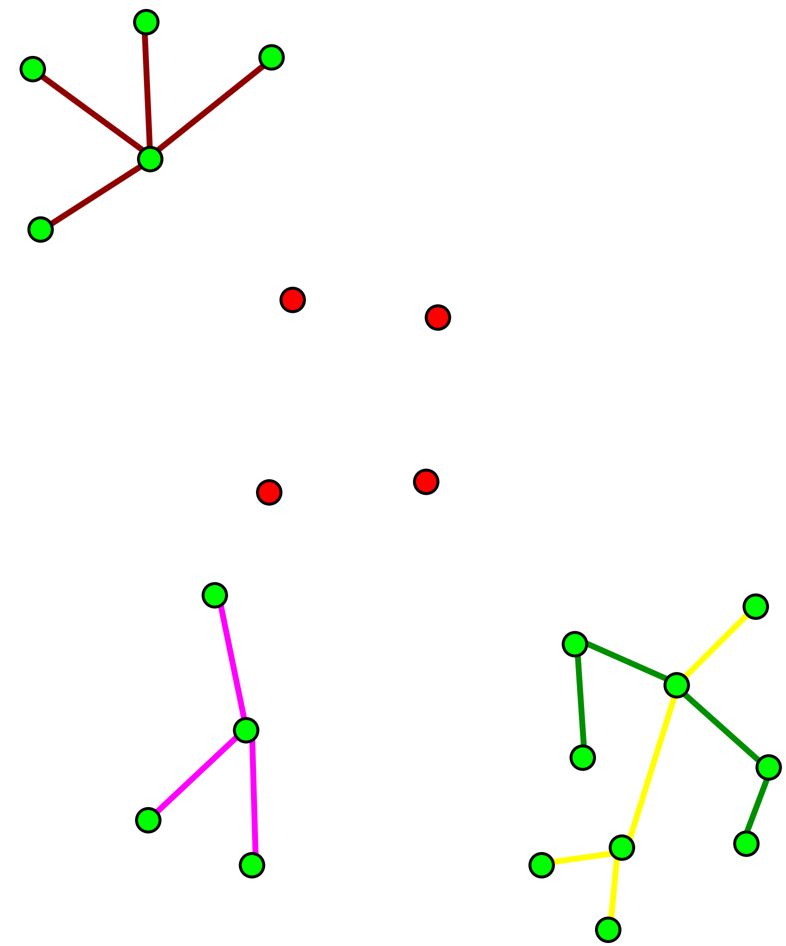

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Input: graph, roots, and B - “guess” of opt. cost.

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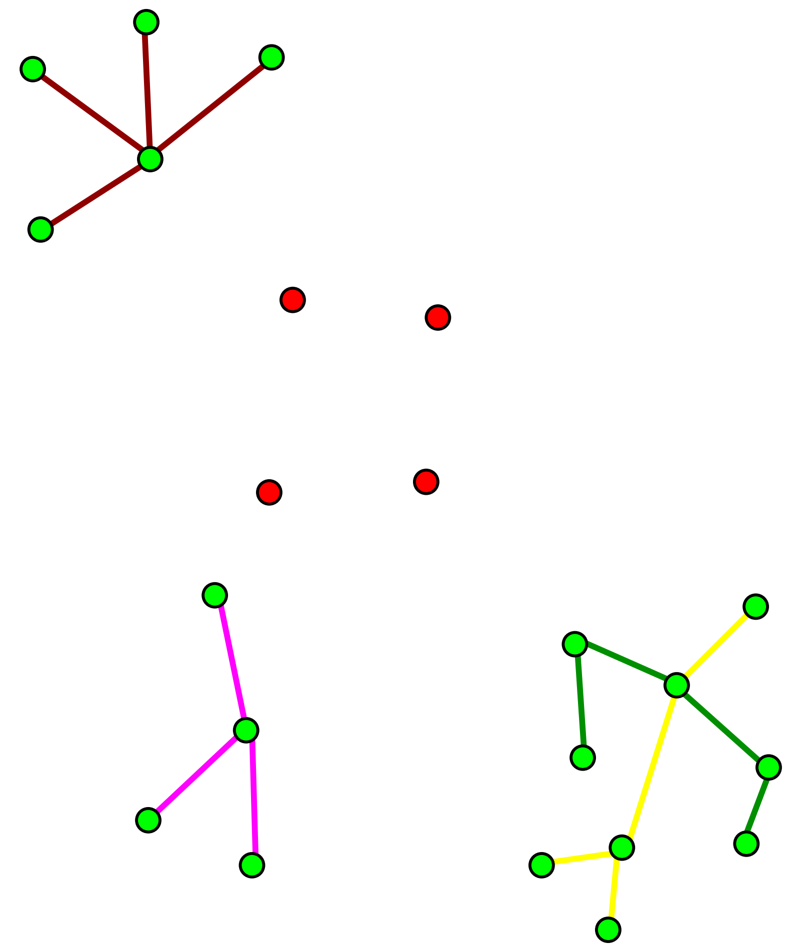

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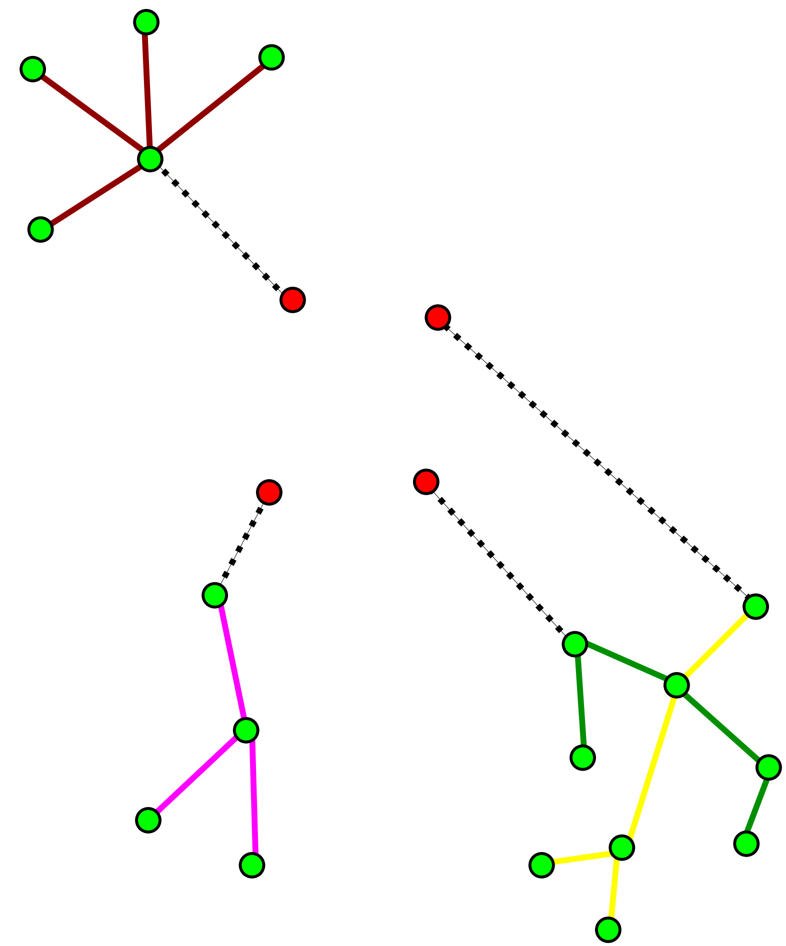

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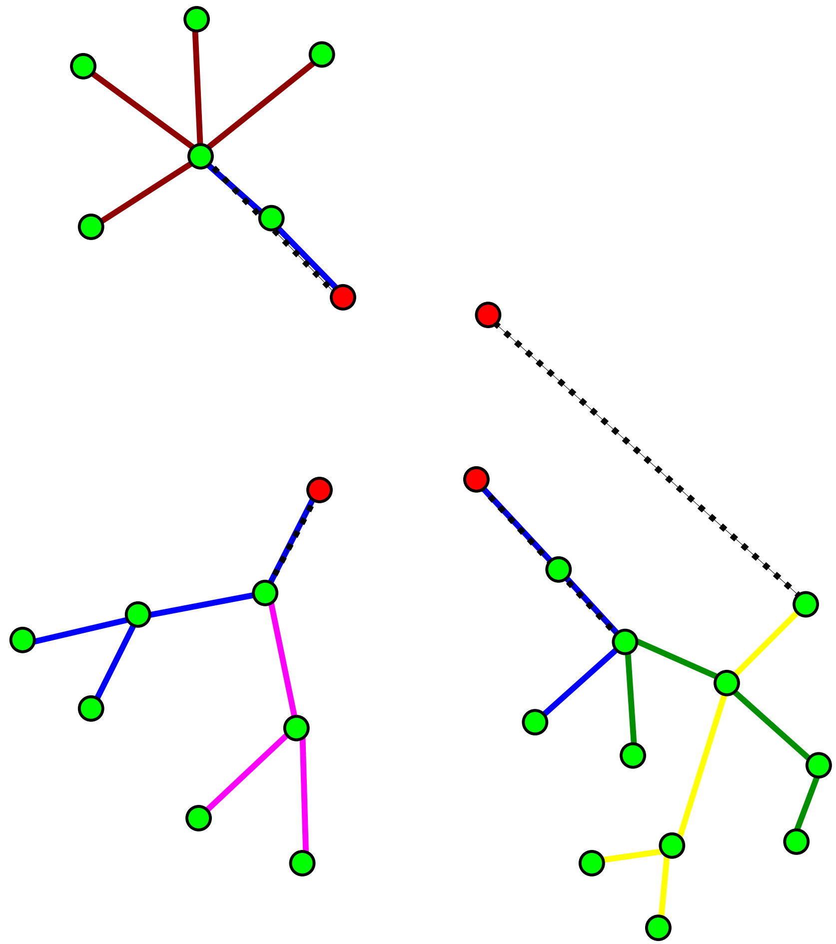
7. else return $\forall r_i$: leftover + matched subtree.



4-approx algorithm : k-rooted tree cover

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Each tree in tree cover may consist of:

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$\Rightarrow$ weight of every tree in solution is $< 4 \cdot B$. QED

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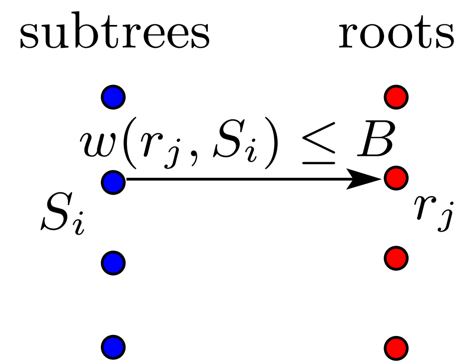

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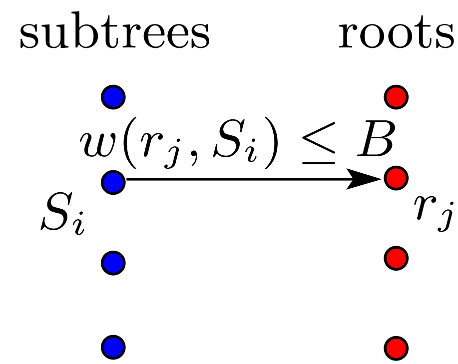

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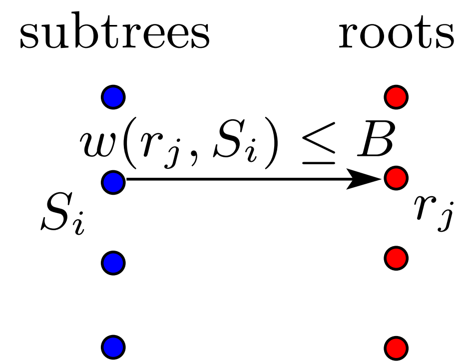

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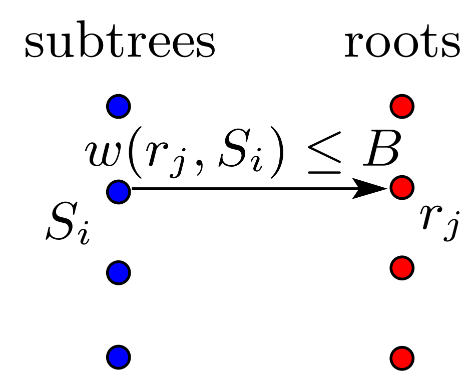

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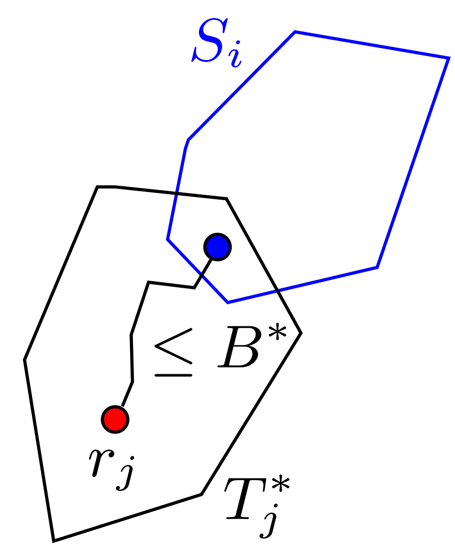

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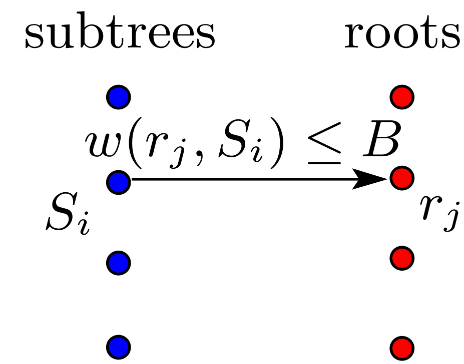

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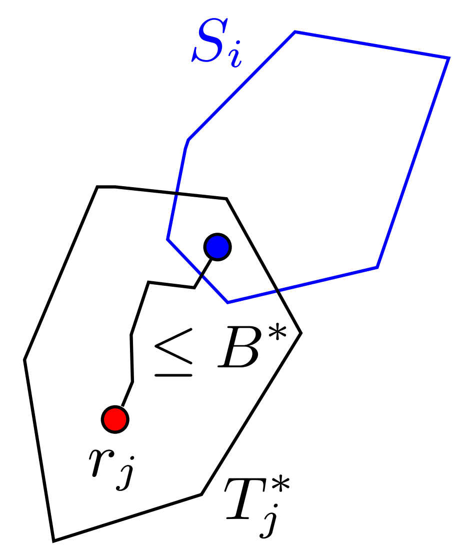

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But

$$T' \triangleq MST - \mathcal{S} + \mathcal{T}^*(\mathcal{S})$$

is a spanning tree and $w(T') < w(MST)$, contradiction. QED

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1. Prune edges $w_e > B$.
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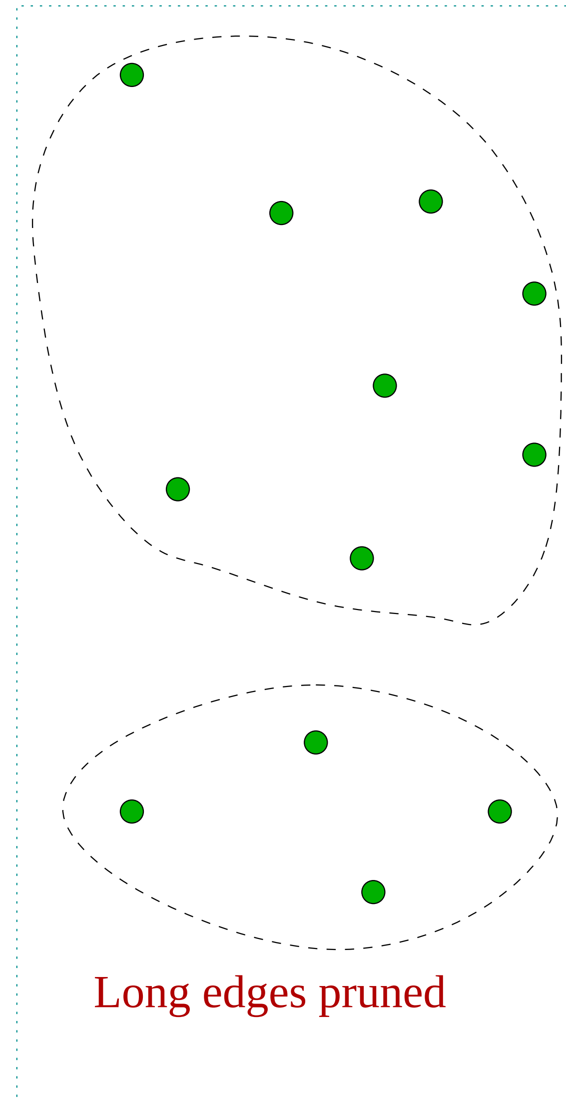

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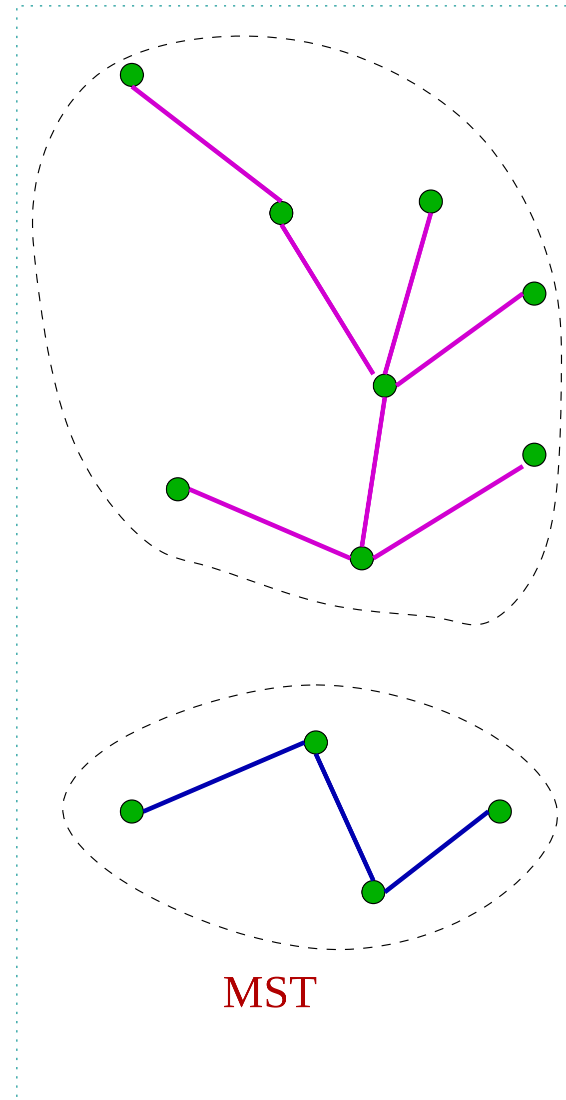

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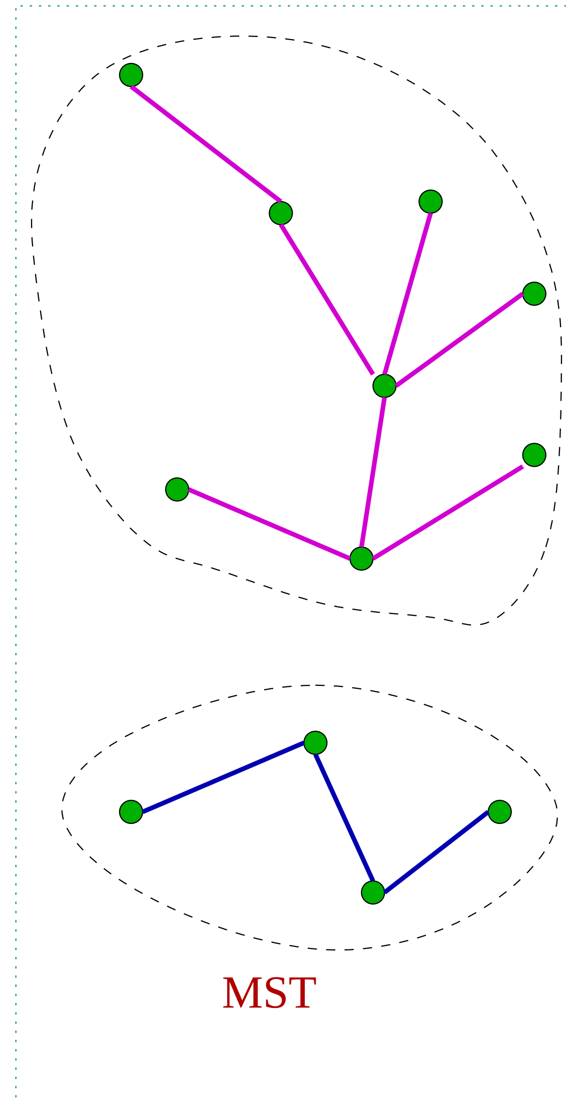

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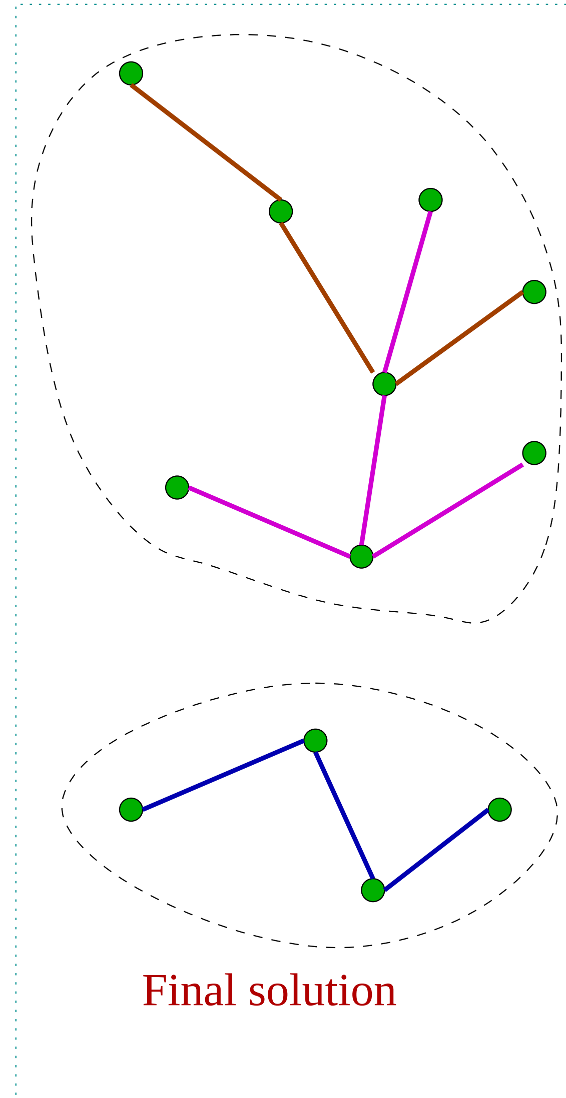

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Alternatively, if $B^* \leq B$, then $k_i + 1 \leq k_i^*$ for all i.

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$$\Rightarrow k_i^* \ge \frac{w(MST_i)}{2B} + \frac{1}{2} > k_i.$$

$\square$

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- Open: rooted path/walk cover.