Fast division: concluding the unfinished symphony of computer arithmetic

Guy Even

SEE-NERGIA talk

Division of Floating Point Numbers

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- Main difficulty in computing f_1 / f_2.

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- Other improvements (increase radix) still require linear time. Used in many microprocessors!
- Faster algorithms can be obtained based on Newton iterations.

processor	latency				
	ALU	FP add	FP mult	FP div single	FP div double
ULTRA-Sparc 3	1	4(1)	4(1)	17(15)	20(18)
Pentium 3	1	3(1)	5(2)	17(17)	32(32)
Pentium 4	1	5(1)	7(2)	23(23)	38(38)
Itanium	1	5(1)	5(1)	30+(11)*	40+(13)*
AMD Athlon	1	4(1)	4(1)	16(13)	20(17)
Power3	1	4(1)	4(1)	17(13)	21(17)
Motorola G4	1	5(1)	5(1)	21(21)	35(35)
Alpha 21064	1	4(1)	4(1)	34(34)	63(63)
Alpha 21164	1	4(1)	4(1)	19(19)	31(31)
Alpha 21264/21364	1	4(1)	4(1)	12(9)	15(12)
R8000	1	4(1)	4(1)	14(11)	20(17)
R12000	1	2(1)	2(1)	14(12)	21(19)
Proposed Divider (1/2)	-	-	-	9(6)	11(8)
Proposed Divider (1)	-	-	-	9(3)	11(4)
Proposed Divider (2)	-	-	-	9(1)	11(2)
Proposed Divider (3)	-	-	-	9(1)	11(1)

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  Root of $f(x) = 0$ is $\frac{1}{B}$.
- Newton iterations: an initial estimate $x_0 \neq 0$ and iterate

$$\begin{aligned} x_{i+1} &= x_i - \frac{f(x_i)}{f'(x_i)} \\ &= x_i - \frac{B - 1/x_i}{1/x_i^2} \\ &= x_i - B \cdot x_i^2 + x_i \\ &= x_i \cdot (2 - B \cdot x_i). \end{aligned}$$

Reciprocal Computation $1/B$ with Newton iterations

Initial estimate x_0, and $x_{i+1} = x_i \cdot (2 - B \cdot x_i)$.

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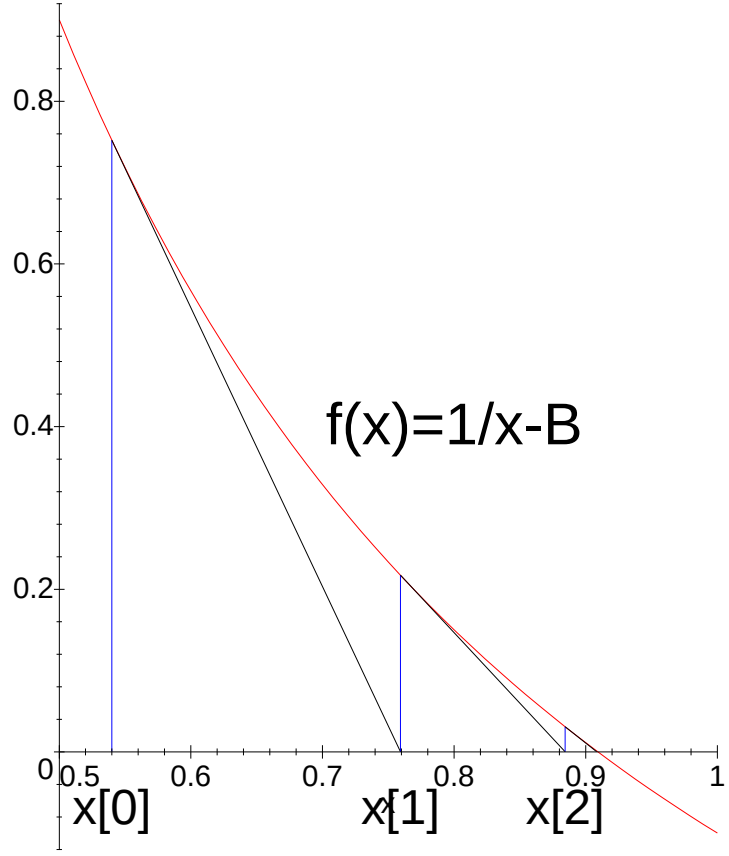

Error analysis : Newton iterations

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- It follows that

$$\begin{aligned} e_{i+1} &= 1 - B \cdot x_{i+1} \\ &= 1 - B \cdot x_i \cdot (2 - B \cdot x_i) \\ &= (1 - B \cdot x_i)^2 = e_i^2. \end{aligned}$$

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  2. Quadratic convergence rate: number of accurate bits doubles in every iteration $\Rightarrow$ after $\log n$ iterations we have n bits of the reciprocal.
  3. $e_{i+1} \geq 0$ implies $x_{i+1} \leq \frac{1}{B}$ (one sided convergence).

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  - Not hard to analyze error since algorithm “recovers” from errors!

Pipelining : Newton iterations

each step requires 3 operations:

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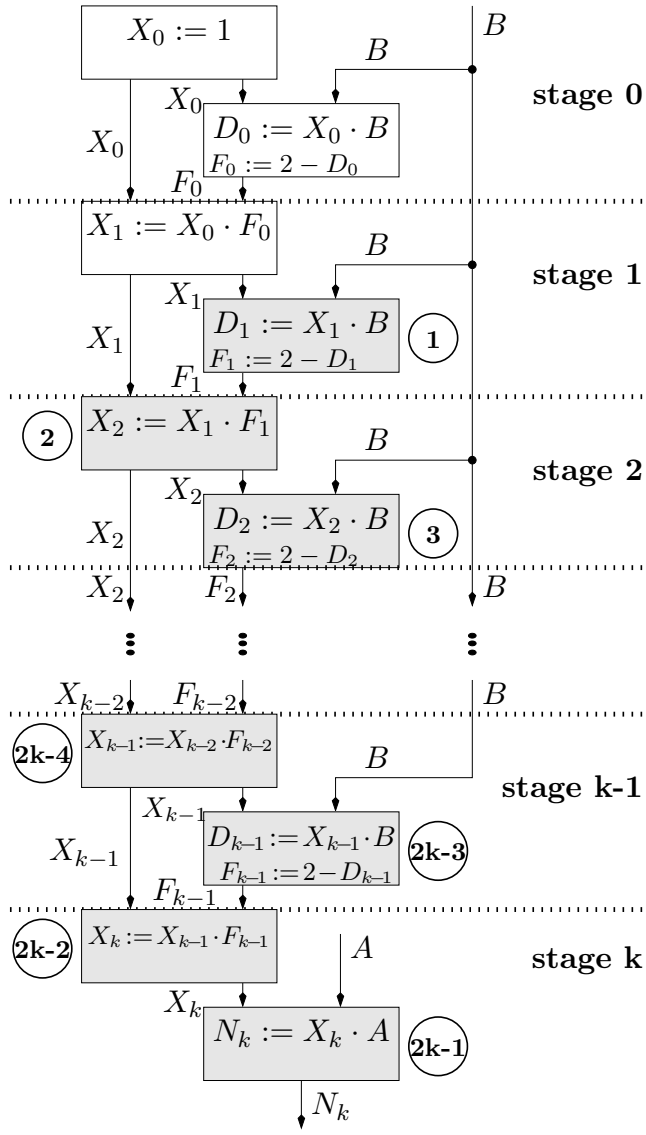

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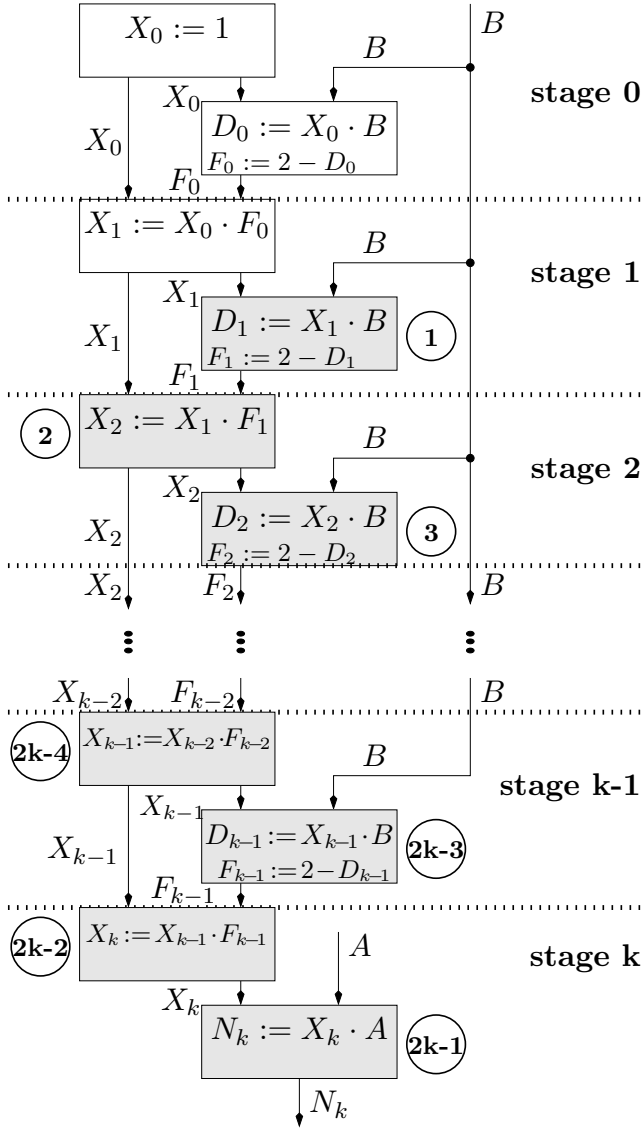
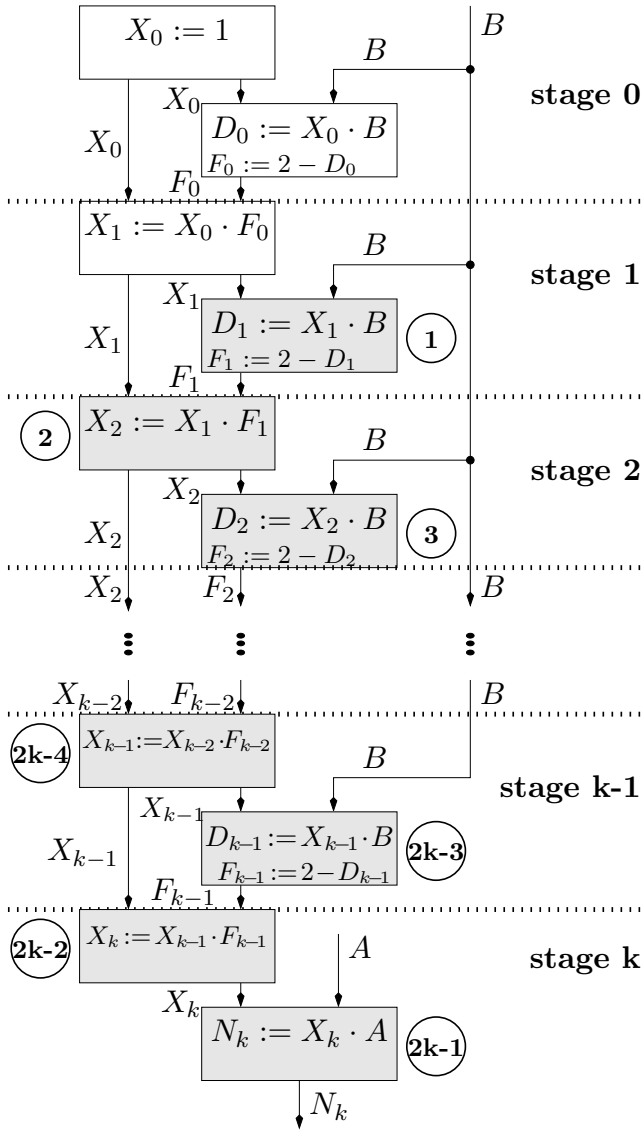
2 dependent multiplications per iteration... slows down computation.



Recap - division with Newton

- $\log n$ iterations compute $1/B$.
- Each iteration requires 2 *dependent* multiplications.
- $\Rightarrow O(\log^2 n)$ delay for computing $1/B$.
- final multiplication $A \cdot (1/B)$ gives quotient.



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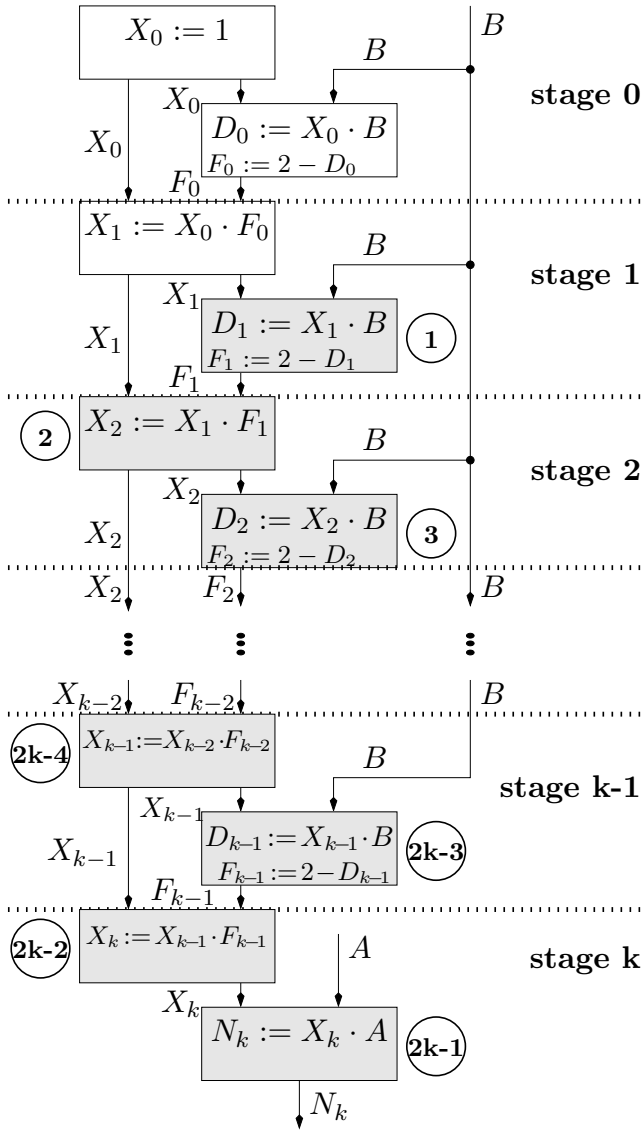
Q: parallelize/pipeline multiplications in each iteration?



Enabling parallelization

Newton iterations

$$(*) \quad x_{i+1} = x_i \cdot F_i,$$

where

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Define

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Multiply both sides of $(*)$ by A & B:

$$\begin{cases} A \cdot x_{i+1} = A \cdot x_i \cdot F_i \\ B \cdot x_{i+1} = B \cdot x_i \cdot F_i \end{cases}$$

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$$\Leftrightarrow$$

$$\begin{cases} N_{i+1} = N_i \cdot F_i \\ D_{i+1} = D_i \cdot F_i \end{cases}$$

Goldschmidt - properties

Since

$$N_i \stackrel{\triangle}{=} A \cdot x_i$$

$$D_i \stackrel{\triangle}{=} B \cdot x_i$$

$$x_i \rightarrow 1/B,$$

it follows that

$$N_i \rightarrow A/B$$

$$D_i \rightarrow 1.$$

Convergence rate - same as Newton iterations! (only if intermediate computations are precise)

Goldschmidt's algorithm - listing

Require: $|e_0| < 1$.

1: Initialize:

$$N_{-1} := A$$

$$D_{-1} := B$$

$$F_{-1} := \frac{1 - e_0}{B}.$$

2: **for** $i = 0$ to k **do**

3: $\quad N_i := N_{i-1} \cdot F_{i-1}$.

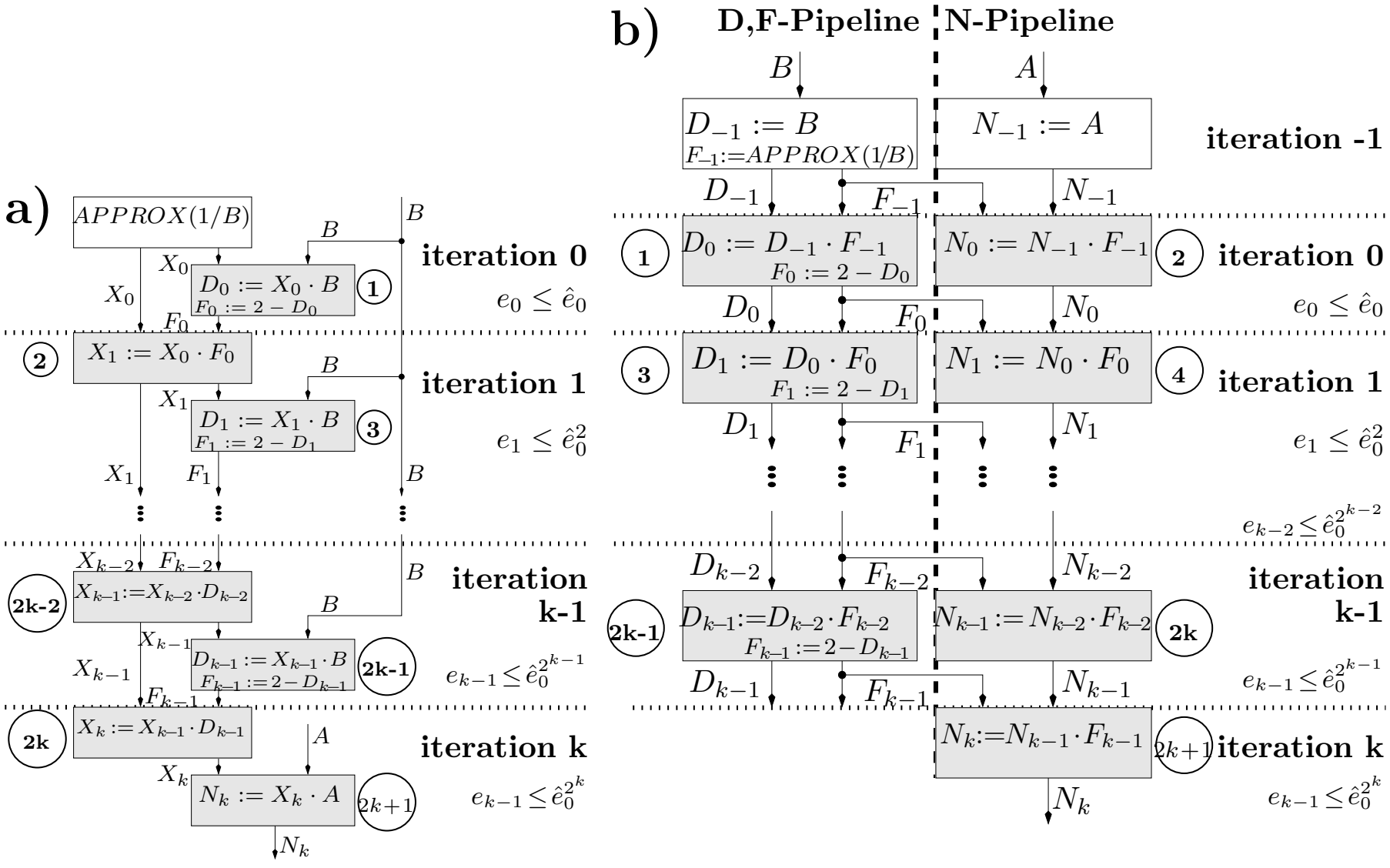
4: $\quad D_i := D_{i-1} \cdot F_{i-1}$.

5: $\quad F_i := 2 - D_i$.

6: **end for**

7: Return(N_i)

Parallelization



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Convergence based on invariant

$$\frac{N_i}{D_i} = \frac{A}{B}.$$

Imprecise computations violate invariant, and

$$N_i \not\to A/B.$$

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- Bounding error used to be complicated:
  - IBM 360 model 91 [1967]: ad hoc error analysis.
  - AMD K7 [1999]: combining formal proof methods that span thousands of lines with millions of test vectors.
- Pessimistic bounds imply larger multipliers that waste area, power, and increased delay.

Our contribution [E+Seidel+Ferguson]

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- Greatly simplify task of verification.

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A complete description of an FP-DIV micro-architecture for single & double precision.

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- Fewer cycles!